CHAPTER # 2

RIGID BODY DYNAMICS
And
ROTATIONAL INERTIA

Rigid Body

A rigid body is defined as a collection of particles whose relative distances are constrained to remain absolutely fixed.

Such bodies do not ~~exist~~ excist in nature because the ultimate components (the atoms) are always undergoing some relative motion. This motion, however is of a microscopic nature and usually may be ignored for the purposes of describing the macroscopic motion of the body. On the other hand, macroscopic displacement (such as deformations) within the body can also take place. For many bodies of intersest we can safely neglect the changes in size and shape due to such deformations and obtain equations of motion which are valid to a high degree of accuracy

There is also a relativistic limitation to the concept of an absolutely rigid body. e.g. Consider a ~~bar~~ long bar of

some material. If we strike a blow at one end of the bar and if the bar were absolutely rigid the effect would be felt instantaneously at the opposite end. But this corresponds to the transmission of a signal with an infinite velocity and this situation is not possible according to relativity theory. Actually the velocity of transmission of such a signal in metal bar is low compared with velocity of light ($\sim 10^5$ cm/sec) and depends on the elastic properties of the material.

OR

A rigid body is considered to be an aggregate of particles bound together by forces of cohesion and internal mutual attractions which are in all cases equal and opposite.

Remarks #. We can use interchangeably the idealized concept of a rigid body as a collection of discrete particles or as a continuous distribution of matter.

Degrees of Freedom and independent Co-ordinates of a rigid Body

A single mass point has one degree of freedom if its motion is restricted to a straight line or a curve, two degrees of freedom if it is made to move

in a plane or on a curved surface. The mass point moving freely in space has three degrees of freedom.

Two mass points connected by a weightless, rigid rod have five degrees of freedom because the 1st point can be regarded as freely moving in which case the 2nd is restricted to the surface of a sphere described about the 1st, its radius equal to the length of the rod.

The number of degrees of freedom for n mass points which are coupled by r relations between their co-ordinates is

$$f = 3n - r$$

If there is an infinity of mass points connected by infinitely many conditions such an enumeration is of course not feasible.

Degrees of Freedom of Freely Moving Body

: We single out a point of rigid body. It has three degrees of freedom. A 2nd point at a constant distance from 1st (Definition of rigid body) can move only on a spherical surface about the 1st point at centre. This gives two more degrees of freedom. Finally a third point can describe a circle about the axis connecting the 1st two points, thus contributes one degree of freedom. Once the motion of these three points have been specified the paths of all other points

of the rigid body are uniquely determined. It follows that

$$f = 3+2+1 = 6$$

Degrees of Freedom of a Top on a Plane Surface

We assume that the bottom of the spinning top terminates in a point and take this as the 1st point of enumeration. It has two degrees of freedom. A second point can move on a hemisphere about the first and a third on on a circle about a line connecting the first two. Thus

$$f = 2+2+1 = 5.$$

Top with Fixed Point

Now the two degrees of freedom of the 1st point are lost so that

$$f = 2+1 = 3$$

Rigid Body with Fixed Axis

Here

$$f = 1$$

If the centre of mass of the body does not lie on the axis we speak of physical or compound pendulum. From this we obtain a mathematical or simple pendulum if the body shrinks to a point.

The spherical pendulum — a mass point restricted to move on the surface of a sphere—

$f = 2$

Infinitely Many Degrees of Freedom.

For a deformable solid or a liquid

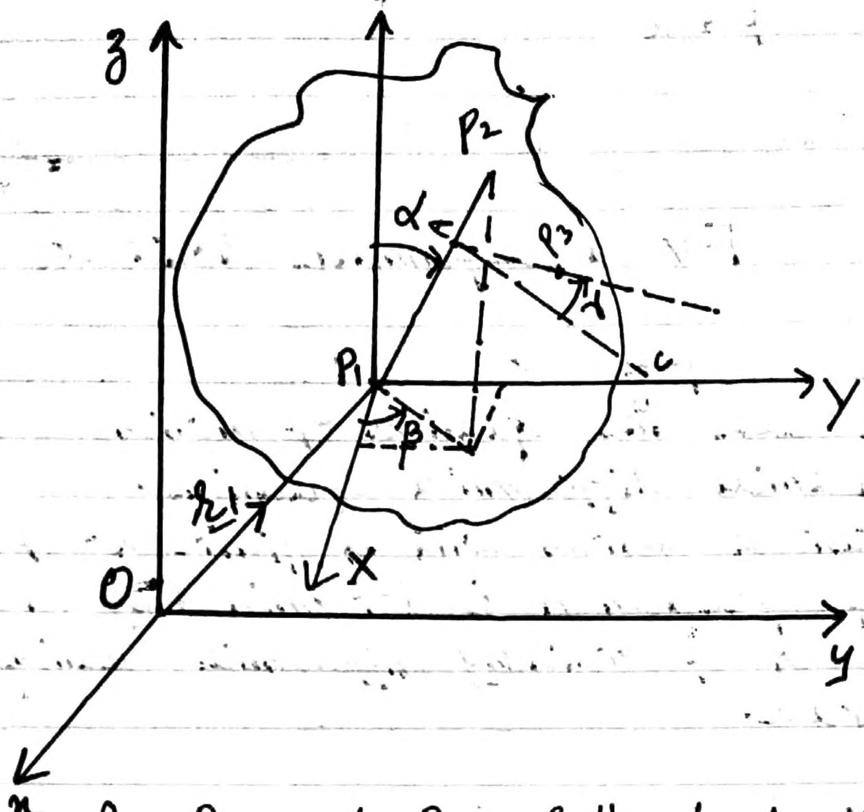
$f = \infty$

In this case the equations of motion become partial differential equations.
By contrast a system with a finite no of degrees of freedom n is determined by an equal no n of ordinary differential equations of 2nd order.

Specification of a Rigid Body

The configuration of a system of particles with respect to an arbitrary set of axes in space can be specified by stating the positions of all the particles. This means that, if there are n particles, then $3n$ co-ordinates are required to locate the system completely. This situation becomes simpler if the particles comprise the rigid body. In this case the position of a rigid body be completely specified by a much smaller number of co-ordinates which are exactly six

Suppose B is a rigid body whose position we want to specify w.r.t the system of axes $oxyz$. To do this it is necessary to select three non-collinear points



P_1, P_2 and P_3 of the body. The Co-ordinates of P_1 in $Oxyz$ system are x_1, y_1, z_1 and provide three of the six Co-ordinates necessary to describe Completely the position of the body. The remaining three, usually the angles, are introduced by stating the orientation of body about P_1: Let P_1X, P_1Y P_1Z be drawn through P_1 and parallel to Ox, Oy, Oz respectively. Draw a line P_1P_2. Orientation of P_1P_2 is obtained by stating angles α, β as shown i.e. P_2 moves on a sphere of radius P_1P_2 and has two degrees of freedom i.e. two variable α, β. The angle α is the angle which P_1P_2 makes with P_1Z and β is the angle which the projection of P_1P_2 in XY-plane makes with P_1X.

The information is not yet sufficient to specify the body B Completely. Since it may rotate about P_1P_2 as an axis without

changing x_1, y_1, z_1, α or β. Third step is to use 3rd point P_3; we draw line P_3A perpendicular to P_1P_2 and lying in the $Z P_1P_2$ plane. The angle γ between AC and and AP_2 supplies the final information about the orientation of the body about axis P_1P_2. Hence the six co-ordinates required to specify the position of the rigid body B are $x_1, y_1, z_1, \alpha, \beta, \gamma$. It will be seen later that the angles α, β, γ are not the angles usually chosen but are merely a convenience explaination at this point.

Plane Motion of Rigid Body # :

A rigid body perform plane motion or motion in two dimensions if all points of the body move in parallel planes. For convenience we generally consider the plane of motion to be the plane which contains centre of mass and we treat the body as a thin slab (or lamina) whose motion is confined to the plane of slab (lamina). Plane motion may translation or Rotation or both)

In case of rotation about a fixed axis, all particles in the rigid body move in circular paths about the axis of rotation and all lines in the body (including those that do not pass through the axis) rotate through the same angle in the same time i.e. have same angular velocity.

Remarks# A rigid body moving in two dimensions has three degrees of freedom and therefor requires three independent Co-ordinates to specify its position and Consequently three equations are necessary to determine its motion.

In a particular case when a rigid body is free to rotate about a fixed axis, its position is completely specified by a single Co-ordinate, which is taken to be angle θ which some plane through axis of rotation, fixed in body, makes with some other plane, fixed in space through the same axis. In such cases, the angular velocity and acceleration at any instant, is the same for every point of the body.

Instantaneous Axis of Rotation

A body turning about a fixed point O does not in general continue to turn about the same axis but the position of axis varies both in body and space.

The position of axis of rotation at any instant can be found if the directions of motion of any two lines OA, OB in the body are known. For if AA′, BB′ represent infinitesimal displacements, then the axis of rotation must lie in a plane through OA perpendicular to AA′ and also in plane through OB perpendicular

to BB'. It is therefore the intersection of these two planes.

The locus of instantaneous axis of rotation in body is a cone (called body cone) and its locus in space is another cone called space cone. The two cones cut a unit sphere whose centre is at their common vertex.

The continuous motion of body about a fixed point can be produced by rolling of the body cone on the space cone. We shall consider later the special case in which these cones are right circular.

General Motion of a Rigid Body#

The general motion of a rigid includes both rotational and translational components e.g in case of a wheel on a moving bicycle. A point P on the rim of such a wheel moves in circle according to an observer in the same reference as the wheel (The rider for instance) but an other oberes fixed to ground would describe the motion in a different manner.

The most general finite displacement of a rigid body is composed of a pure translation plus a rotation about a suitably chosen point.

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Rotation and Translation of Rigid Body.

Here we give another explainotion about rotation and traslation of a rigid.

A rigid body can be regarded as made up of a series of parallel Laminate sections.

Consider a body which is moving so that

(1) It centre of mass is moving in a straight line

(2) each of the constituent laminas is rotating in its own plane

(3) The axis of rotation passes through the centre of mass of each lamina section and hence through the centre of mass of the body.

Then each lamina moves as through it were rotating under the action of its own resultant torque about the common axis through the centres of mass. Also because the body is rigid, every Lamina has the same angular acceleration. Thus the whole body moves as though it were rotating under the action the total resultant torque about an axis through the centre of mass.

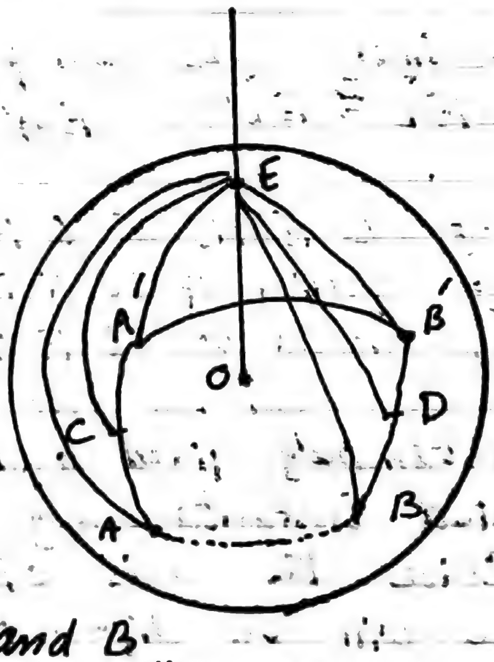
The motion of such a body of mass M can therefore be analysed in the same way as the motion of a lamina moving in its own plane i.e by investigating seperately

(i) the linear motion of a particle of mass M at the centre of mass.
(ii) the rotation of the body about an axis through the centre of mass.

Euler's Theorem

Statement # It states that if one point in a rigid body is kept fixed in space, then an arbitrary displacement of the body is simply a rotation about some axis passing through the fixed point.

Proof #



Since the shape and size of rigid body is not given, therefore we may take a spherical rigid body with centre O fixed. Let A and B be two fixed points on the sphere. Because O is fixed in space, therefore when body moves points A & B moves such that they remain at some great circle of the spherical rigid body. Let A', B' be the new positions of the points A & B after an infinitesimal displacement. Join A, B A' B' by the arcs of great circle

Let C, D be mid points of arcs $\widehat{AA'}$ and $\widehat{BB'}$ respectively. Through C, D draw axes at right angles, which meet the point E on the sphere. Jocining E with A, B, A', B' by great circle arcs, Consider the spherical triangles ΔEAC & $\Delta EA'C$.

Then

$$m\angle ECA = m\angle ECA' \quad \text{(right angles)}$$

$$m\widehat{AC} = m\widehat{CA'} \quad \text{C is mid pt}$$

$$EC = EC \quad \text{Common}$$

So

$$\Delta EAC \cong \Delta EA'C$$

$$\Rightarrow \widehat{EA} \cong \widehat{EA'} \quad \text{(By definition of congruent triangles)}$$

Similarly we can prove that

$$\widehat{EB} \cong \widehat{EB'} \qquad \widehat{EA'} \cong \widehat{EA} \qquad \widehat{AB} \cong \widehat{A'B'} \quad \text{(rigid Body condition)}$$

$$\Rightarrow \Delta EAB \cong \Delta EA'B'$$

$\Rightarrow$ portion of the rigid body lying in ΔEAB has moved to $\Delta EA'B'$.

In the above process E is instantaneously fixed and therefore body has rotated instantaneously about an axis OE.

Since the axis OE is not fixed, therefore there will be different axes of rotation passing through O.

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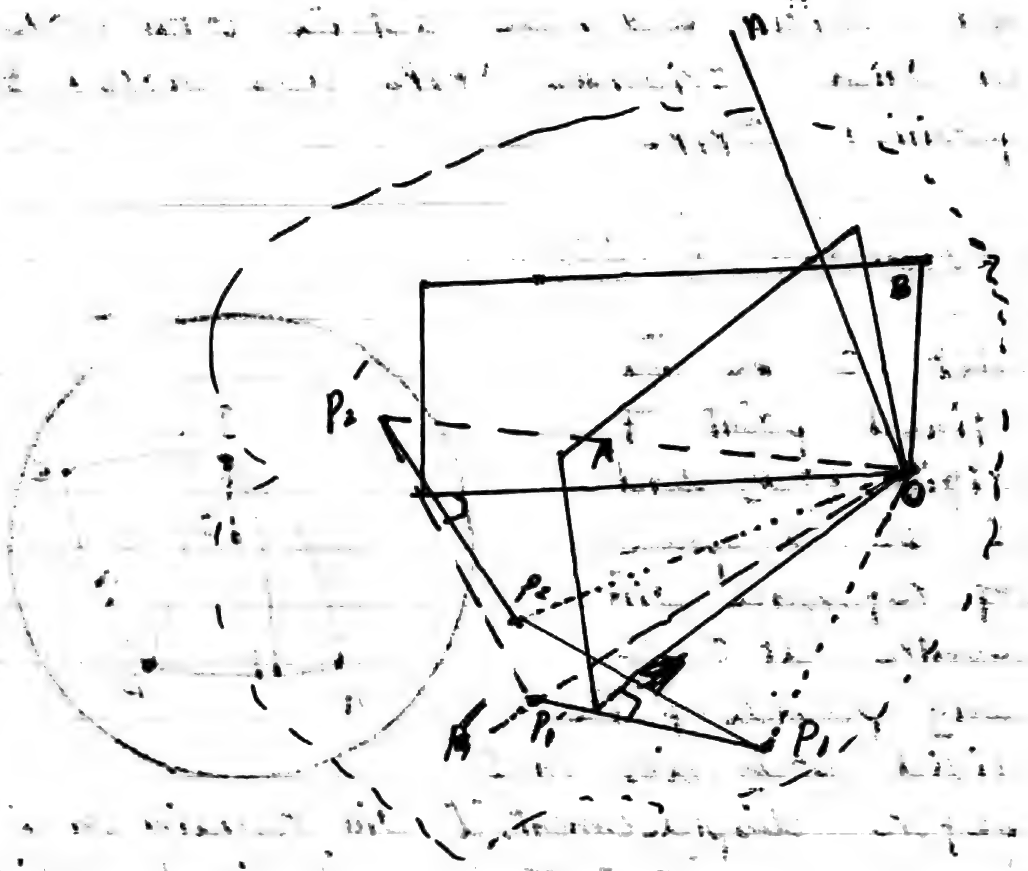
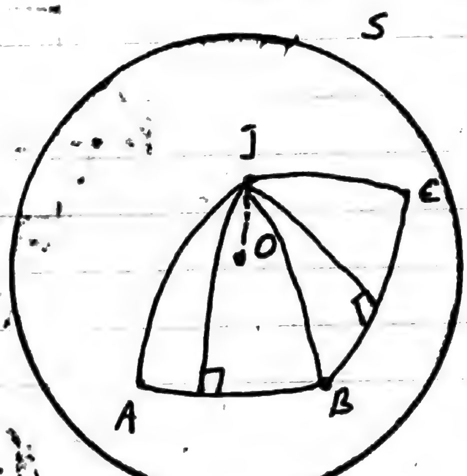
Another Proof of Euler's Theorem *



Suppose O is the fixed point of rigid body and P_1, P_2 are any other two points in the body. After an arbitrary finite motion about O the triangle OP_1P_2 has moved to position $OP_1'P_2'$. Line OP_1 may revolve to its new position OP_1' by rotating about any axis which passes through O and also lies in plane A which bisects the angle P_1OP_1' and which is normal to the plane P_1OP_1'. Similarly line OP_2 may revolve about any axis which passes through O and which also lies in a plane B which bisects the angle P_2OP_2' and which is normal to the plane P_2OP_2'. Now the intersection On of these two planes is the

unique rotation axis which is such that when the body is given single rotation through an angle $\Delta\theta$, we get the total movement in which triangle OP_1P_2 has moved to position OP_1P_2

Another Proof



Let O be the fixed point of rigid body and S be a sphere of reference with centre at O. If any particle of the rigid body lies on S before displacement, it will remain on S because point O of body is fixed. Consider a particle at A and another particle at B before displacement. Suppose the particle at A moves to B and particle at B moves to C. Draw great circles arcs $\widehat{AB}$, $\widehat{BC}$ on S and also draw the great circles which bisect these arcs at right angles. Let J be the one of two points of intersections of these great circles. Join points A, B, C to J by arcs of great circles. By construction arcs AJ, BJ, CJ are equal.

Also $\widehat{AB} = \widehat{BC}$ (by rigidity of body)

Then the spherical triangles ABJ & BCJ are equal in all respects. Therefore a rotation about JO through an angle AJB takes A to B and B to C, leaves O fixed. Thus this

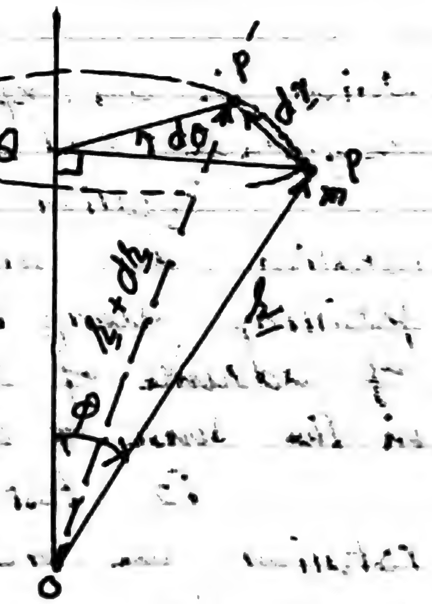
rotation moves three non-collinear particles originally at O, A, B to the positions O, B, C and is equivalent to this displacement considered.

Infinitesimal Rotation

Problem # Define infinitesimal rotation and prove that two infinitesimal rotations follow the parallelogram of vector addition. Also prove that velocity $\underline{v}$ of any particle of a rigid body rotating ~~about~~ with one point fixed with instantaneous angular velocity $\underline{\omega}$ is given by

$$\underline{v} = \underline{\omega} \times \underline{r}$$

where $\underline{r}$ is radius vector ~~for~~ from fixed point of body



Sol # suppose a rigid body with point O fixed has OQ instantaneous axis of rotation and that very instant particle m is at point P with position vector $\underline{r}$ from O. During infinitesimal time interval dt, the body rotates through an infinitesimal angle $d\theta$. The particle m will also rotate through angle $d\theta$ along a circular arc because body is rigid. Suppose particle is displaced through an infinitesimal displacement $d\underline{r}$ pointing perpendicular to the plane of OPQ. Thus

position of particle after time dt is $\underline{r} + d\underline{r}$.

Now the radius of circular arc along which particle is displaced is

$$QP = r \sin\phi$$

and the magnitude of infinitesimal vector $d\underline{r}$ is given by

$$|d\underline{r}| = r \sin\phi \, d\theta \rightarrow ①$$

(by $S = r\theta$ because chord $PP' \equiv$ arc PP' in infinitesimal rotation.)

Since the direction of $d\underline{r}$ is perpendicular to OQP and its magnitude given by ① is cross-product of vector $\underline{r}$ with a vector of magnitude $d\theta$ pointing along OQ, axis of rotation. Thus if we take unit vector $\underline{n}$ along $\overrightarrow{OQ}$, then we may write

$$d\underline{r} = \underline{n}\, d\theta \times \underline{r}$$

$$= \underline{d\theta} \times \underline{r} \qquad \rightarrow ②$$

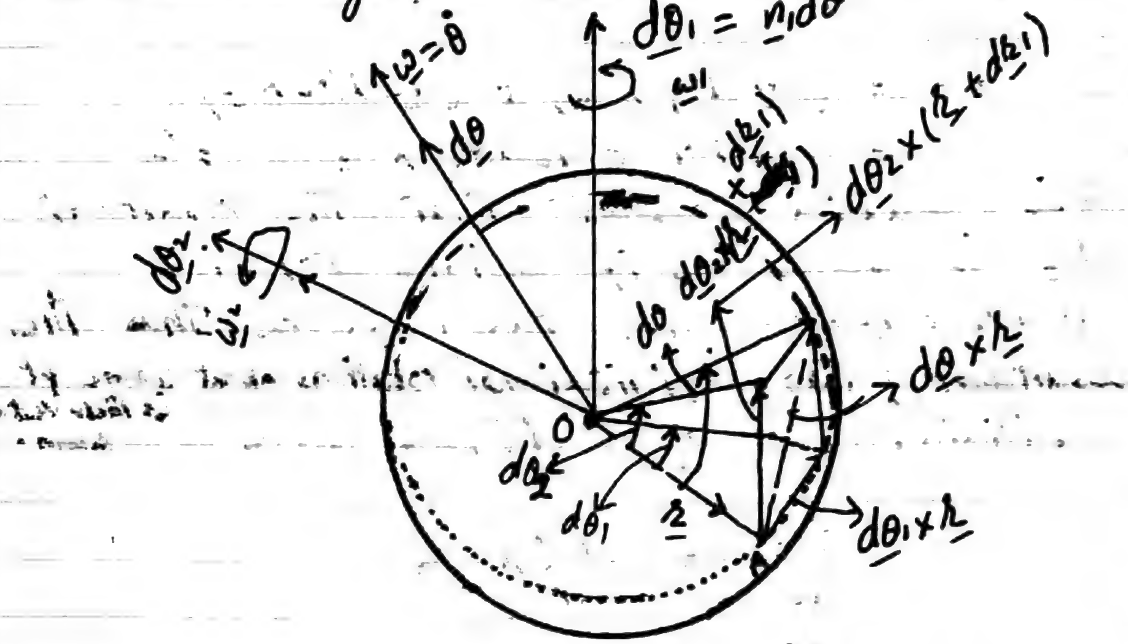
which is an expression giving $d\underline{r}$ both in magnitude and direction.

Equation ② suggests that an infinitesimal rotation $d\theta$ may be represented by a vector pointing along the axis of rotation in the direction of advance of a right-handed screw turning in the sense of $d\theta$.

To test the vector notation for rotation we need to confirm whether or not two such infinitesimal rotations may be added vectorially.

Consider two successive infinitesimal rotation $\underline{n}_1 d\theta_1$, $\underline{n}_2 d\theta_2$, in which two axes of rotation intersect at a common point O, where $\underline{n}_1$ $\underline{n}_2$ are unit

Vectors along two axes



After a rotation $\underline{n}_1 d\theta_1$, $\underline{r}$ has suffered an infinitesimal change $d\underline{r}_1$ and p.v of particle A has change to $\underline{r} + d\underline{r}_1$. The rotation $\underline{n}_2 d\theta_2$, then induces a change $d\underline{r}$ into radius vector $\underline{r} + d\underline{r}_1$ and is given by

$$d\underline{r}_2 = \underline{n}_2 d\theta_2 \times (\underline{r} + d\underline{r}_1)$$

Now position vector of particle A after 2nd rotation becomes

$\underline{r} + d\underline{r}_1 + d\underline{r}_2$

$= \underline{r} + (\underline{n}_1 d\theta_1 \times \underline{r}) + \underline{n}_2 d\theta_2 \times (\underline{r} + d\underline{r}_1)$

$= \underline{r} + (\underline{n}_1 d\theta_1 \times \underline{r}) + \underline{n}_2 d\theta_2 \times (\underline{r} + \underline{n}_1 d\theta_1 \times \underline{r})$

The resultant $d\underline{r}$ of $d\underline{r}_1$ & $d\underline{r}_2$ is

$d\underline{r} = d\underline{r}_1 + d\underline{r}_2$

$= (\underline{n}_1 d\theta_1 \times \underline{r}) + [\underline{n}_2 d\theta_2 \times (\underline{r} + d\underline{r}_1)]$

Neglecting 2nd order infinitesimals

$$d\underline{r} = (\underline{n}_1 d\theta_1 \times \underline{r}) + (\underline{n}_2 d\theta_2 \times \underline{r})$$
$$= (\underline{n}_1 d\theta_1 + \underline{n}_2 d\theta_2) \times \underline{r} \longrightarrow ③$$

This equation shows that the resultant $d\underline{r}$ is same as if single rotation $\underline{n}_1 d\theta_1 + \underline{n}_2 d\theta_2$ has taken place. Thus the resultant of two infinitesimal rotations about same pt is their vector sum.

Dividing ② by dt, we have

$$\frac{d\underline{r}}{dt} = \frac{d\theta}{dt} \times \underline{r}$$
$$\underline{v} = \underline{\omega} \times \underline{r}$$

Dividing ③ by dt

$$\frac{d\underline{r}}{dt} = \left(\underline{n}_1 \frac{d\theta_1}{dt} + \underline{n}_2 \frac{d\theta_2}{dt}\right) \times \underline{r}$$
$$\underline{v} = (\underline{\omega}_1 + \underline{\omega}_2) \times \underline{r}$$
$$= \underline{\omega} \times \underline{r} \longrightarrow ④$$

where $\underline{\omega} = \underline{\omega}_1 + \underline{\omega}_2$

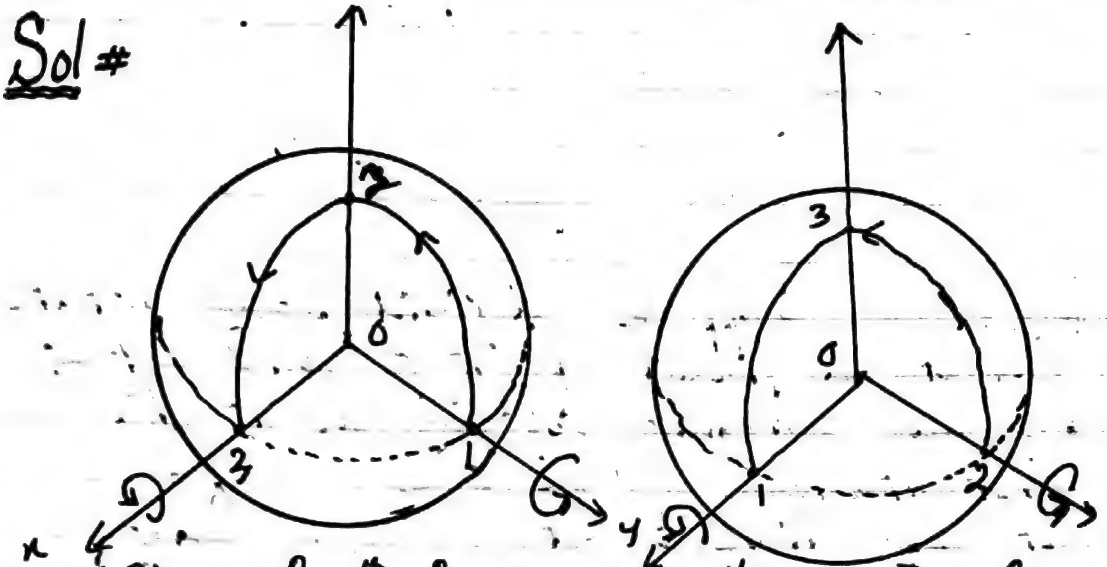
This equation gives the information that the angular velocity which is obtained by dividing instantaneous rotation by dt also obey the parallelogram law of vector addition

If referred to a set of rectangular co-ordinate axes, the angular velocity $\underline{\omega}$ like other may be expressed in terms of components along these axes as

$$\underline{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

Problem # Prove that Finite rotations of a rigid body are not Commutative.

Sol #



(a) θ_x, then θ_y (b) θ_y Then θ_x

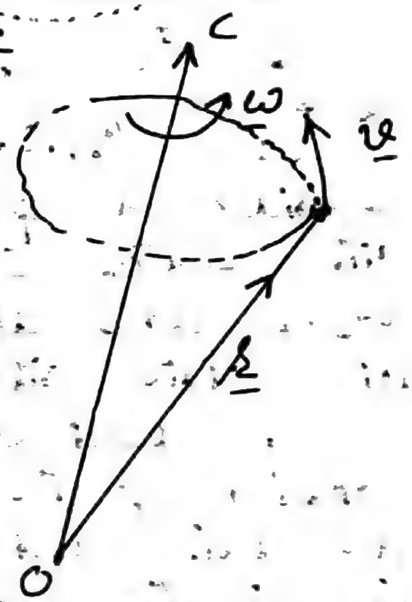
Consider a sphere cut from the rigid body with centre O as fixed point. The x-y-z axes are fixed in space so do not rotate with the body.

In part (a) two successive 90° deg rotation of the sphere about, first, the x-axis and second, the y-axis result in in the motion of a point which is initially on the y-axis in position 1, to positions 2 and 3 respectively. On the other hand if the order of rotation is reversed, then the point on the y-axis at position 2 suffers no motion during the y-rotation but moves to position 3 during 90°-deg rotation about the x-axis. Thus two cases do not yield the same final position. Thus finite rotations do not Commute.

Problem # Let $\underline{\omega}$ be the instantaneous angular velocity of a rigid body rotating about a fixed point and $\underline{V}$ be velocity of any point P fixed in the body, then prove that

$$\underline{\omega} = \frac{1}{2}\, \text{Curl}\, \underline{V}.$$

Sol # Suppose a rigid body rotate about a fixed point O and let $\underline{\omega}$ be the angular velocity about the instantaneous axis OC at any time t. Let P be a point fixed in the body such that



$$\overrightarrow{OP} = \underline{r}$$

Then linear velocity $\underline{v}$ of point P is given by

$$\underline{v} = \underline{\omega} \times \underline{r}$$

In Cartesian form let

$$\underline{\omega} = [\omega_1, \omega_2, \omega_3] \qquad \underline{r} = [x, y, z]$$

$$\underline{V} = \underline{\omega} \times \underline{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= (\omega_2 z - \omega_3 y)\hat{i} + (\omega_3 x - \omega_1 z)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}$$

$$\Rightarrow \text{Curl } \underline{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= 2\omega_1 \hat{i} + 2\omega_2 \hat{j} + 2\omega_3 \hat{k}$$

$$= 2(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k})$$

$$= 2\,\underline{\omega}$$

$$\Rightarrow \boxed{\underline{\omega} = \frac{1}{2}\text{ Curl } \underline{V}}$$ Proved

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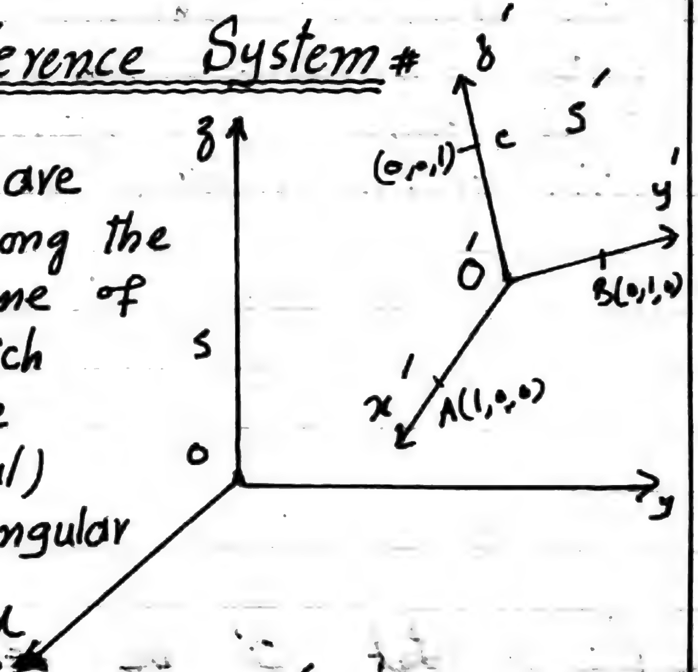
Time Derivatives of Unit Vectors of Rotating Reference System#



suppose $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the axes of a frame of reference S' which rotates w.r.t the Newtonian (inertial) frame S with angular velocity Ω.
Then $\hat{i}, \hat{j}, \hat{k}$ also rotate with frame S' are not constant. Therefore their time derivatives

are required.

We may think $\hat{i}$ as position vector of a point $A(1,0,0)$ relative to base point O'. Then $\frac{d\hat{i}}{dt}$ is velocity of A relative to O' and we have

$$\frac{d\hat{i}}{dt} = \underline{\Omega} \times \hat{i} \quad (\because \underline{V} = \underline{\omega} \times \underline{r})$$

Similarly considering $\hat{j}$ & $\hat{k}$ as position vectors of points $B(0,1,0)$ & $C(0,0,1)$ we have

$$\frac{d\hat{j}}{dt} = \underline{\Omega} \times \hat{j} \qquad \frac{d\hat{k}}{dt} = \underline{\Omega} \times \hat{k}$$

Problem # Prove that the if a rigid body is rotating about a fixed point and $\underline{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$ is instantaneous angular velocity about an axis through the fixed, where $\hat{i}, \hat{j}, \hat{k}$ are unit orthogonal vectors along axes of a frame of reference fixed in body with origin at fixed point, then

$$\frac{d\hat{i}}{dt} = \omega_3\hat{j} - \omega_2\hat{k}$$

$$\frac{d\hat{j}}{dt} = \omega_1\hat{k} - \omega_3\hat{i}$$

$$\frac{d\hat{k}}{dt} = \omega_2\hat{i} - \omega_1\hat{j}$$

Sol # Let O' be fixed point w.r.t Newtonian frame $Oxyz$ and $O'x'y'z'$ be frame of reference fixed in body

rotating about O' with instaneous angular velocity

$$\underline{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$$

where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along axes of frame $O'x'y'z'$ respectively.

Then

$$\frac{d\hat{i}}{dt} = \underline{\omega} \times \hat{i}$$

$$= (\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) \times \hat{i}$$

$$= \underline{0} - \omega_2 \hat{k} + \omega_3 \hat{j}$$

$$= \omega_3 \hat{j} - \omega_2 \hat{k}$$ proved.

$$\frac{d\hat{j}}{dt} = \underline{\omega} \times \hat{j}$$

$$= (\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) \times \hat{k}$$

$$= \omega_1 \hat{k} + \underline{0} - \omega_3 \hat{i}$$

$$\frac{d\hat{j}}{dt} = \omega_1 \hat{k} - \omega_3 \hat{i}$$ proved

$$\frac{d\hat{k}}{dt} = \underline{\omega} \times \hat{k}$$

$$= (\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) \times \hat{k}$$

$$= -\omega_1 \hat{j} + \omega_2 \hat{i} + \underline{0}$$

$$= \omega_2 \hat{i} - \omega_1 \hat{j}$$ Proved.

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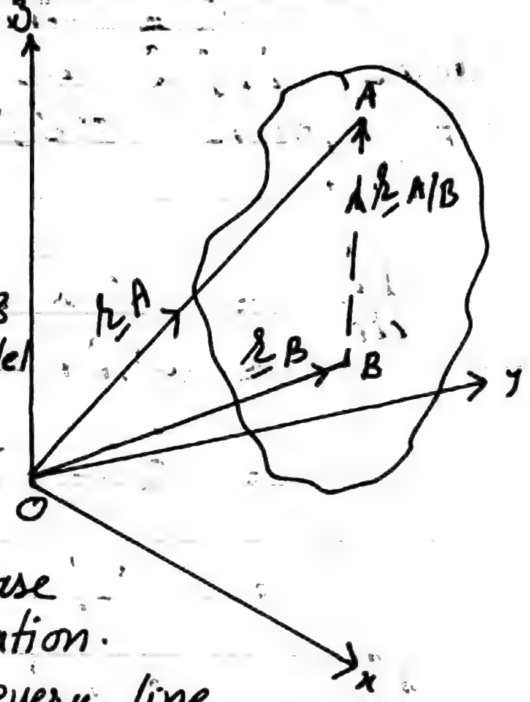
Translation of Rigid Body



Consider a rigid body translating in three dimensional space.
Any two points in the rigid body such as A, B will move along parallel straight lines in case of rectilinear translation or will move along congruent curves in case of curvilinear translation.
In either case every line in the body such as AB remain parallel to its original position.

Let $\underset{\sim}{r}_A$, $\underset{\sim}{r}_B$ be p.v. of A & B w.r.t fixed origin O and $\underset{\sim}{r}_{A/B}$ be p.v of A w.r.t B. Then

$$\underset{\sim}{r}_A = \underset{\sim}{r}_{A/B} + \underset{\sim}{r}_B \quad \rightarrow (1)$$

$\because$ Body is rigid

$\therefore$ $\underset{\sim}{r}_{A/B}$ remains constant

Diff (1) w.r.t t

$$\frac{d\underset{\sim}{r}_A}{dt} = \underset{\sim}{0} + \frac{d\underset{\sim}{r}_B}{dt}$$

$$\underset{\sim}{V}_A = \underset{\sim}{V}_B$$

Again differentiating

$$\underset{\sim}{a}_A = \underset{\sim}{a}_B$$

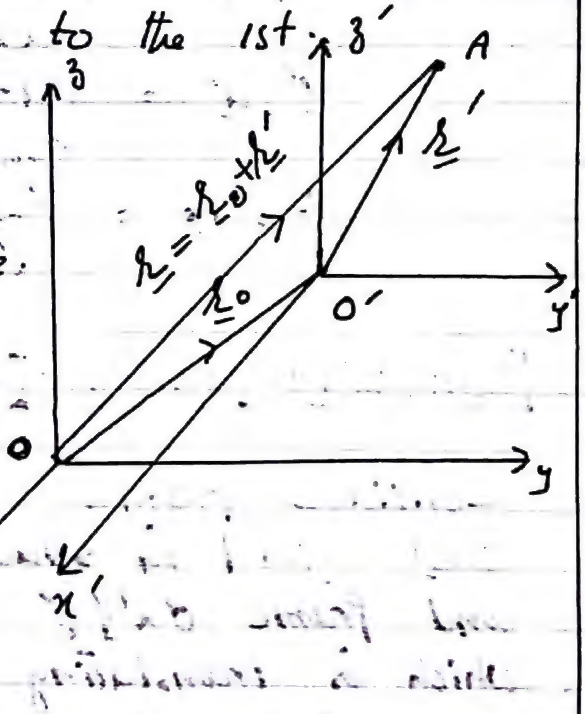
Thus all points in the body have the same velocity and accelration.

Moving Reference Axes and Relative Motion

To completely understand the general motion of a rigid body it is necessary to first understand the relative motion of a particle in moving frames of reference. A frame of reference may have only translatory or only rotatory motion or more general motion consisting of translation and rotation.

Translating Referenc Axes

We often need to connect the velocities (accelerations) of a particle relative to two different frames of reference one static and the other moving relative to the 1st.



Lets: $oxyz$ be fixed frame S': $o'x'y'z'$ be (movi) translating frame. Let at any time t a particle A has p.v $\overrightarrow{O'A} = \underline{r}'$ relative to S' and p.v $\overrightarrow{OA} = \underline{r}$ relative to S, Then we have p.v $\underline{r}$ as

$$\underline{r} = \overrightarrow{OO'} + \underline{r}'$$

$$= \underline{r}_0 + \underline{r}' \longrightarrow ①$$

differential changes in p.Vs during infinitesimal time 'dt' will be given as

$$d\underline{r} = d\underline{r}_o + d\underline{r}' \rightarrow ②$$

$\Rightarrow$ differential change in p.v. $\underline{r}$ of A w.r.t to origin o is equal to the sum of differential change of p.v of o' w.r.t o and p.v. $\underline{r}'$ of A relative to o'.

Dividing ② by dt

$$\frac{d\underline{r}}{dt} = \frac{d\underline{r}_o}{dt} + \frac{d\underline{r}'}{dt}$$

$$\underline{V} = \underline{V}_o + \underline{V}'$$

$$\text{\&} \quad \frac{d\underline{V}}{dt} = \frac{d\underline{V}_o}{dt} + \frac{d\underline{V}'}{dt}$$

$$\underline{a} = \underline{a}_o + \underline{a}'$$

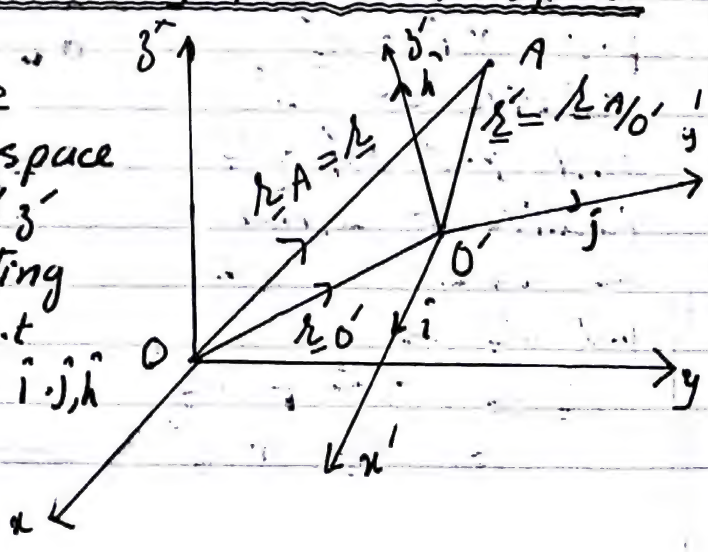
where $\underline{V}, \underline{a}$ = Velocity and acceleration relative to S

$\underline{V}', \underline{a}'$ = Velocity and acceleration of A relative to S'

$\underline{V}_o, \underline{a}_o$ = Velocity and acceleration of S' relative to

Translating and Rotating frames of Refence

Consider frame oxyz fixed in space and frame $o'x'y'z'$ which is translating and rotating w.r.t fixed frame. Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors.



along the axes of moving frame. Suppose the moving frame rotates with angular velocity ω instaneously. Consider the space motion of a particle A as observed from rotating system and fixed system.

P.V equation is

$$\underline{r}_A = \underline{r}_{A/o'} + \underline{r}_{o'}$$

$$\underline{r} = \underline{r}' + \underline{r}_{o'} \rightarrow ①$$

where $\underline{r}$ gives the absolute position of A i.e p.v relative to fixed system

$\underline{r}'$ is P.V of A relative to O'

$\underline{r}_{o'}$ is P.V of O' relative to O

Now $\underline{r}' = x'\hat{i} + y'\hat{j} + z'\hat{k}$

so ① is

$$\underline{r} = \underline{r}_{o'} + x'\hat{i} + y'\hat{j} + z'\hat{k}$$

Diff w.r.t t

$$\frac{d\underline{r}}{dt} = \frac{d\underline{r}_{o'}}{dt} + (\dot{x}'\hat{i} + \dot{y}'\hat{j} + \dot{z}'\hat{k})$$
$$+ x'\frac{d\hat{i}}{dt} + y'\frac{d\hat{j}}{dt} + z'\frac{d\hat{k}}{dt}$$

But $\dfrac{d\hat{i}}{dt} = \underline{\omega} \times \hat{i}$ $\quad \dfrac{d\hat{j}}{dt} = \underline{\omega} \times \hat{j}$

$\dfrac{d\hat{k}}{dt} = \underline{\omega} \times \hat{k}$

$$\Rightarrow \frac{d\underline{r}}{dt} = \frac{d\underline{r}_{o'}}{dt} + (\dot{x}'\hat{i} + \dot{y}'\hat{j} + \dot{z}'\hat{k})$$
$$+ x'\underline{\omega} \times \hat{i} + y'\underline{\omega} \times \hat{j} + z'\underline{\omega} \times \hat{k}$$

$$\frac{d\underline{r}}{dt} = \frac{d\underline{r}_{o'}}{dt} + \dot{x}'\hat{i} + \dot{y}'\hat{j} + \dot{z}'\hat{k} + \underline{\omega} \times (x'\hat{i} + y'\hat{j} + z'\hat{k})$$

$$\underline{V}_A = \underline{V}_{o'} + \dot{x}'\hat{i} + \dot{y}'\hat{j} + \dot{z}'\hat{k} + \underline{\omega} \times \underline{r}'$$

Now the observer in moving frame measures velocity components $\dot{x}', \dot{y}', \dot{z}'$, so

$\dot{x}'\hat{i} + \dot{y}'\hat{j} + \dot{z}'\hat{k} = \underline{V}_{rel}$ which is the velocity of A relative to moving frame.

Thus the relative velocity equation is

$$\underline{V}_A = \underline{V}_{o'} + \underline{\omega} \times \underline{r}' + \underline{V}_{rel}$$

$$= \underline{V}_{o'} + \underline{V}_{rel} + \underline{\omega} \times \underline{r}' \rightarrow ②$$

$\cancel{\underline{V}_{rel} = \underline{V}_{A/o'}}$

If the moving system were non-rotating, then $\underline{\omega}$ woul be absent and we would have relative-velocity equation as

$$\underline{V}_A = \underline{V}_{o'} + \underline{V}_{A/o'} \rightarrow ③$$

Comparing ② & ③, we note that

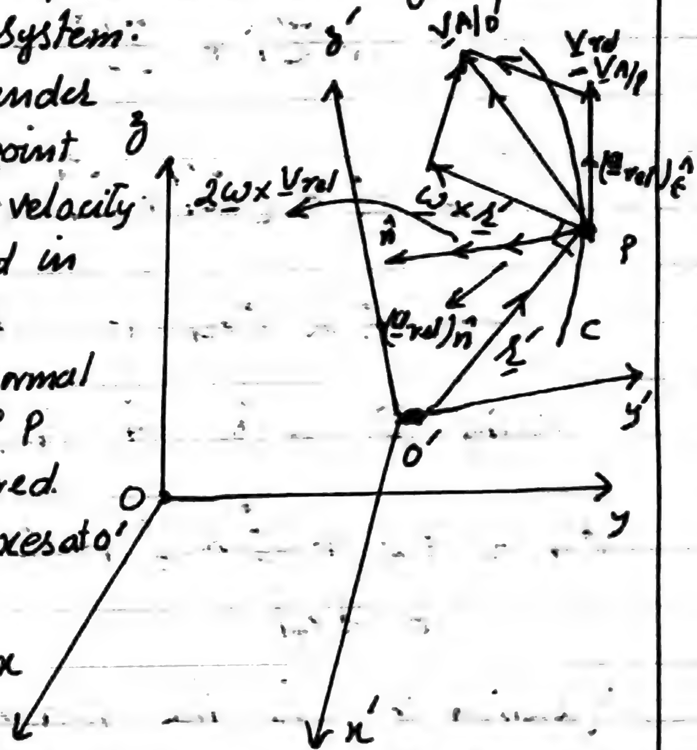
$$\underline{V}_{A/o'} = \underline{\omega} \times \underline{r}' + \underline{V}_{rel}$$

$$\Rightarrow \underline{\omega} \times \underline{r}' = \underline{V}_{A/o'} - \underline{V}_{rel}$$

$\Rightarrow$ $\underline{\omega} \times \underline{r}'$ is the difference between the relative velocities as measured from rotating and non-rotating axes.

Explaination of Terms $\underline{\omega}\times\underline{r}'$ & $\underline{V}_{rel}$

The term $\underline{\omega}\times\underline{r}' + \underline{V}_{rel}$ gives the velocity of A relative to O′ as measured from non-rotating axes or axes fixed in space. To explain these terms further we consider motion of particle A along curve C in rotating system.



Let the moment under consider A is at point P. $\underline{V}_{A/P} = \underline{V}_{rel}$ is velocity of A as measured in O′x′y′z′ frame.

$\underline{\omega}\times\underline{r}' = \underline{V}_{P/O'}$ is normal to $\underline{r}'$ & is velocity of P ~~relative to O′~~ measured from non-rotating axes at O′

So $\underline{V}_P = \underline{V}_{O'} + \underline{V}_{P/O'}$ is the absolute velocity of P which represents the effect of moving co-ordinate system both translational and rotational. Thus $\underline{V}_{A/O'} = \underline{\omega}\times\underline{r}' + \underline{V}_{rel}$ is relative velocity of A relative to O′ as measured from non-rotating system at O′

Relative acceleration equation may be obtained by differentiating the relative velocity equation ②

$$\underline{a}_A = \underline{a}_{O'} + \underline{\dot{\omega}}\times\underline{r}' + \underline{\omega}\times\underline{\dot{r}}' + \underline{\dot{V}}_{rel} \rightarrow ④$$

$$\text{Now } \underline{\omega}\times\underline{\dot{r}}' = \underline{\omega}\times\frac{d}{dt}(x'\hat{i} + y\hat{j} + z\hat{k})$$

$$= \underline{\omega}\times(\underline{\omega}\times\underline{r}') + \underline{\omega}\times\underline{V}_{rel} \rightarrow ⑤$$

$$\dot{\underline{V}}_{rel} = \frac{d}{dt}(\dot{x}'\hat{i} + \dot{y}'\hat{j} + \dot{z}'\hat{k})$$

$$= \dot{x}'\frac{d\hat{i}}{dt} + \dot{y}'\frac{d\hat{j}}{dt} + \dot{z}'\frac{d\hat{k}}{dt} + \ddot{x}'\hat{i} + \ddot{y}'\hat{j} + \ddot{z}'\hat{k}$$

$$= \dot{x}'\,\underline{\omega}\times\hat{i} + \dot{y}'\,\underline{\omega}\times\hat{j} + \dot{z}'\,\underline{\omega}\times\hat{k} + \ddot{x}'\hat{i} + \ddot{y}'\hat{j} + \ddot{z}'\hat{k}$$

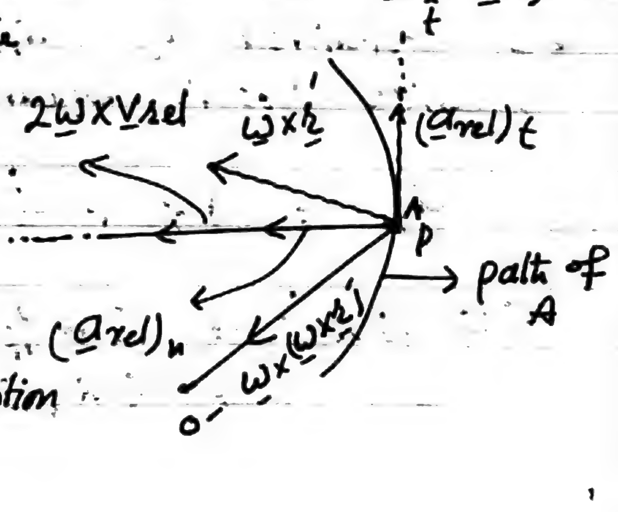
$$= \underline{\omega}\times(\dot{x}'\hat{i} + \dot{y}'\hat{j} + \dot{z}'\hat{k}) + (\ddot{x}'\hat{i} + \ddot{y}'\hat{j} + \ddot{z}'\hat{k})$$

$$= \underline{\omega}\times\underline{V}_{rel} + \underline{a}_{rel} \quad \to (6)$$

using (5) & (6) in (4), we have

$$\underline{a}_A = \underline{a}_{O'} + \dot{\underline{\omega}}\times\underline{r}' + \underline{\omega}\times(\underline{\omega}\times\underline{r}) + 2\underline{\omega}\times\underline{v}_{rel} + \underline{a}_{rel} \quad \to (7)$$

Equation (7) is the general equation for the absolute acceleration (acc measured w.r.t. fixed frame) of a particle A in terms of its acceleration $\underline{a}_{rel}$ measured relative to moving co-ordinate system which rotates with angular velocity $\underline{\omega}$.
The terms $\dot{\underline{\omega}}\times\underline{r}'$ and $\underline{\omega}\times(\underline{\omega}\times\underline{r}')$ represent respectively the tangential ~~normal~~ and normal ~~tangential~~ components of acc $\underline{a}_{P/O'}$ of the coincident point P in its circular motion relative to O'. This motion



will be observed from a set of non-rotating system attached with O' and moving with O'.

The magnitude of $\dot{\underline{\omega}} \times \underline{r}'$ is $r'\ddot{\theta}$ and its direction is tangent to the circle. The magnitude of $\underline{\omega} \times (\underline{\omega} \times \underline{r}') = r'\omega^2$ and its direction is from P to O' along the normal to the circle. $\underline{a}_{rel}$ is acceleration of A measured from the rotating frame of reference $O'x'y'z'$. $\underline{a}_{rel}$ may be expressed in rectangular, normal and tangential or polar co-ordinates in the rotating system. Generally n- and t- components are used.

The term $2\underline{\omega} \times \underline{V}_{rel}$ is called Coriolis (name of French military engineer G-Coriolis) acceleration. It represents the difference between the acc of A relative P as measured from non-rotating axes and from rotating axes. The direction is always normal to the vector $\underline{V}_{rel}$ and the sense is established by right hand rule for cross-product.

The following comparison will clarify the difference between the relative acceleration equations written for rotating and non-rotating axes

$$\underline{a}_A = \underline{a}_{O'} + \underbrace{\dot{\underline{\omega}} \times \underline{r}' + \underline{\omega} \times (\underline{\omega} \times \underline{r}')} + \underbrace{2\underline{\omega} \times \underline{V}_{rel} + \underline{a}_{rel}}$$

$$= \underbrace{\underline{a}_{O'} + \underline{a}_{P/O'}} + \underline{a}_{A/P}$$

$$= \underline{a}_P + \underbrace{\underline{a}_{A/P}}$$

$$= \underline{a}_{O'} + \underline{a}_{A/O'}$$

Here $\underline{a}_{P/o'}$ = acc of P relative to o' measured from non-rotating system at o'

$\underline{a}_{A/P}$ = acc of A relative to P as measured from non rotating frame of reference

$\underline{a}_{rel}$ = acc of A relative to P as measured in a rotating system.

$\underline{a}_{A/o'}$ = acc of A w.r.t. o' measured in a non-rotating system.

Velocity Equation can also be similarly written as

$$\underline{V}_A = \underline{V}_{o'} + \underline{\omega} \times \underline{r}' + \underline{V}_{rel}$$

$$= \underbrace{\underline{V}_{o'} + \underline{V}_{P/o'}} + \underline{V}_{A/P}$$

$$= \underline{V}_P + \underbrace{\underline{V}_{A/P}}$$

$$= \underline{V}_{o'} + \underline{V}_{A/o'}$$

$\underline{V}_{P/o'}$ = Vel of P relative to o' measured from non-rotating system

$\underline{V}_{A/P}$ = $\underline{V}_{rel}$ = Vel of A as measured from rotating system

$\underline{V}_P$ = Absolute vel of P and represents the effect of moving axes both translational and rotational

$\underline{V}_{A/o'}$ = vel of A measured from non-rotating system at o'

Remarks # (1) from

$$\underline{V}_A = \underline{V}_{o'} + \underline{V}_{P/o'} + \underline{V}_{A/P}$$

$$= \underline{V}_{o'} + \underline{V}_{A/o'}$$

where $\underline{V}_{A/o'}$ = velocity of A measured from non-rotating system at o'.

$$= \underline{V}_{P/o'} + \underline{V}_{A/P}$$

$$\left(\frac{d\underline{r}}{dt}\right)_{fixed} = \underline{\omega} \times \underline{r}' + \left(\frac{d\underline{r}'}{dt}\right)_{rotating}$$

where $\left(\frac{d\underline{r}'}{dt}\right)_{rotating}$ represents time derivative of vector $\underline{r}'$ expressed in rotating system as measured in the rotating system and $\left(\frac{d\underline{r}'}{dt}\right)_{fixed}$ is time derivative of $\underline{r}'$ as measured in a fixed non rotating system. Thus we have a rule for transformation of time derivative of P.V between rotating and non-rotating axes. In operator form we have

$$\left(\frac{d}{dt}\right)_{fixed} = \left(\frac{d}{dt}\right)_{rotating} + \underline{\omega} \times$$

This operator can be applied to any vector which changes with time.

(2) # In analysis of acceleration using a rotating frame of reference it is frequently convenient to take the origin of reference co-ordinates at the point P, coincident with the position of particle A at the instant under

Such a choice eliminates the terms $\dot{\underline{\omega}} \times \underline{r}'$ and $\underline{\omega} \times (\underline{\omega} \times \underline{r}')$ since the vector $\underline{r}'$ vanishes and relative acc. equation is

$$\underline{a}_A = \underline{a}_P + 2\underline{\omega} \times \underline{v}_{rel} + \underline{a}_{rel}$$

The terms $\dot{\underline{\omega}} \times \underline{r}'$ and $\underline{\omega} \times (\underline{\omega} \times \underline{r}')$ are submerged in the calculation of acc of P.

When this form is used, it is noted that point P may not be picked at random since it is the one point in the rotating reference system coincident with the particle A at the time of analysis.

Problem # Rotating disk has a radial slot in which a small particle A is confined to slide. Let the disk turn with a constant angular velocity $\omega = \dot{\theta}$ and let the particle move along the slot with a constant speed $v_{rel} = \dot{x}$ relative to slot. Find the magnitude of the Coriolis acceleration and also the net acc $\underline{a}_A$ of A.

Sol #

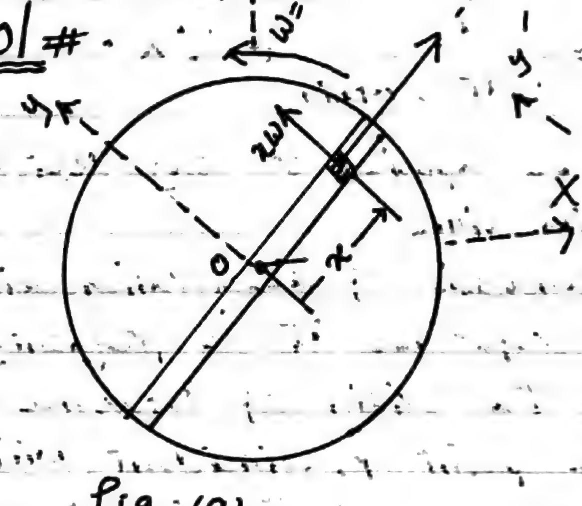


Fig. (a)

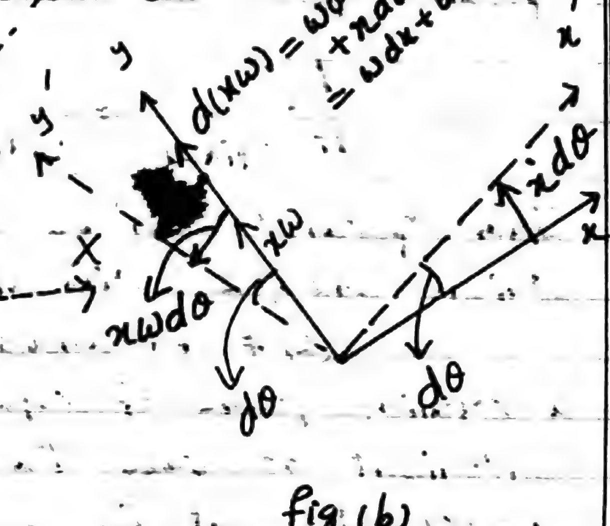


Fig. (b)

The velocity of A has two components.

(a) $\dot{x}$ due to motion along the slot

(b) $x\omega$ due to the rotation of the slot

The changes in these two components are shown in fig (b) for the interval dt during which the x-y axes rotate with disk through the angle $d\theta$ to $x'-y'$.

The velocity increment due to change in direction of $\underline{V}_{rel}$ is $\dot{x}\,d\theta$ and that due to the magnitude of $x\omega$ is $\omega\,dx$, both along the y-axis normal to the slot. Dividing both increment by dt and adding, we have

$$\omega\dot{x} + \dot{x}\omega = 2\dot{x}\omega$$

which is the magnitude of Coriolis acc

$$2\underline{\omega} \times \underline{V}_{rel}$$

The increment due to change in the direction of $x\omega$ is $x\omega\,d\theta$. Dividing it by dt we get $x\omega\dot{\theta} = x\omega^2$, which gives the acceleration of a point P fixed to the slot and momentarily coincident with particle A. Now we check the validity of equation

$$\underline{a}_A = \underline{a}_{o'} + \dot{\underline{\omega}} \times \underline{r}' + \underline{\omega} \times (\underline{\omega} \times \underline{r}') + 2\underline{\omega} \times \underline{V}_{rel} + \underline{a}_{rel}$$

Taking o' at o, fixed centre, we have

$$\underline{a}_{o'} = \underline{o}$$

$$\dot{\underline{\omega}} \times \underline{r}' = \underline{o} \quad \because \omega \text{ is constant}$$

with $\underline{V}_{rel}$ constant in magnitude and no curvature to the slot

$$\underline{a}_{rel} = \underline{o}$$

Thus $\underline{a}_A = \underline{\omega} \times (\underline{\omega} \times \underline{r}') + 2\underline{\omega} \times \underline{V}_{rel}$

$$\underline{r}' = x\hat{i} \qquad \underline{\omega} = \omega\hat{k} \qquad \underline{V}_{rel} = \dot{x}\hat{i}$$

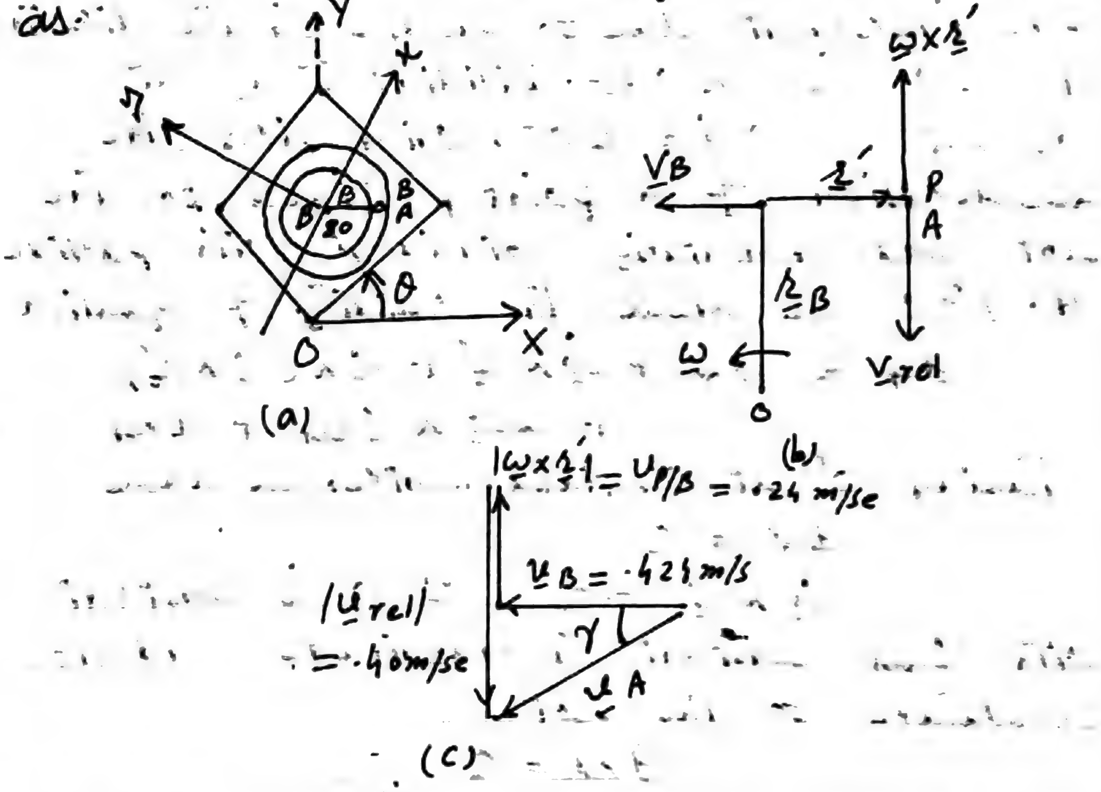
$$\therefore \underline{a}_A = -x\omega^2\hat{i} + 2\dot{x}\omega\hat{j}$$

Problem # Particle A moves in a circular groove of 80mm - radius at the same time grooved plate rotates about its corner O at the rate $\omega = \dot{\theta}$. Determine the absolute velocity of A in the position for which $\theta = 45^\circ$, angle of rotation β of initial position of A at this instant $\beta = 45^\circ$ if at this instant $\dot{\theta} = 3\,rad/s$ and $\dot{\beta} = 5\,rad/s$. side of squar plate is 200 mm

Sol # Axes x-y attached to the plate with origin at B form the rotating axes system. The relative velocity equation is

$$\underline{V}_A = \underline{V}_B + \underline{\omega} \times \underline{r}' + \underline{V}_{rel}$$

The terms shown in fig b, are calculated as



point B moves in a circular arc around O, so its velocity has magnitude

$$|\underline{V}_B| = |\underline{r}_B|\,\omega = (.10\sqrt{2})\,3 = .424\ m/s$$

The term $\underline{\omega} \times \underline{r}'$ is the velocity relative to B of point P coincident with A. The line PB rotates with angular velocity $\omega = \dot{\theta}$ so that

$$|\underline{\omega} \times \underline{r}'| = |\underline{V}_{P/B}| = r\omega$$

$$= .08(3) = 0.24 \text{ m/s}$$

and its direction is up as shown in fig c. The relative velocity $\underline{V}_{rel}$ of A w.r.t plate depends upon $\dot{\beta}$ and has magnitude

$$|\underline{v}_{rel}| = r'\dot{\beta} = (.08)(5) = .40 \text{ m/se}$$

and its direction is down as shown. Adding the three vectors shown in fig (c). we have

$$v_A = \sqrt{(0.424)^2 + (.40 - 0.24)^2}$$

$$= 0.453 \text{ m/s}$$

$$\gamma = \tan^{-1}\left(\frac{.40 - 0.24}{0.424}\right) = 20.7^\circ$$

Rate of change of a Vector

OR

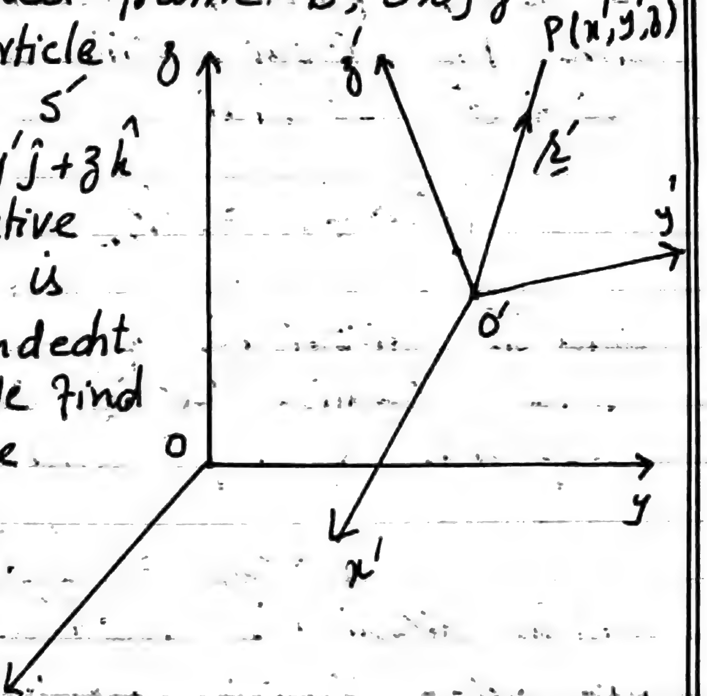
Transformation of time derivative between rotating and non-rotating axes

Problem # For any time dependent vector function, obtain a relationship between fixed and rotating frames #

Sol # Let $\hat{i}, \hat{j}, \hat{k}$ be a triad of unit orthogonal vectors in frame of reference $S': O'x'y'z'$ ...

which rotates with angular velocity $\underline{\omega}$ relative to a fixed frame S; $oxyz$

Let P be a particle moving in frame S' and $\underline{r}' = x'\hat{i} + y'\hat{j} + z\hat{k}$ be p.v of P relative to S'. Now $\underline{r}'$ is surely time dependecht vector function. We find its time derivative w.r.t fixed axes and rotating axes.



Now $\underline{r}' = x'\hat{i} + y'\hat{j} + z\hat{k}$

Taking time derivative of $\underline{r}'$ as measured w.r.t S-system, we have

$$\left(\frac{d\underline{r}'}{dt}\right)_{fixed} = \dot{x}'\hat{i} + \dot{y}'\hat{j} + \dot{z}'\hat{k} + x'\frac{d\hat{i}}{dt} + y'\frac{d\hat{j}}{dt} + z'\frac{d\hat{k}}{dt}$$

$$= \dot{x}'\hat{i} + \dot{y}'\hat{j} + \dot{z}'\hat{k} + x'\underline{\omega}\times\hat{i} + y'\underline{\omega}\times\hat{j} + z'\underline{\omega}\times\hat{k}$$

(Note for an observer in fixed system-S
$\frac{d\hat{i}}{dt} = \underline{\omega}\times\hat{i}$ $\quad \frac{d\hat{j}}{dt} = \underline{\omega}\times\hat{j}$ $\quad \frac{d\hat{k}}{dt} = \underline{\omega}\times\hat{k}$)

$$= \dot{x}'\hat{i} + \dot{y}'\hat{j} + \dot{z}'\hat{k} + \underline{\omega}\times\underline{r}'$$

Now an observer in rotating system components of derivative of $\underline{r}'$ are $\dot{x}', \dot{y}', \dot{z}'$. So $\dot{x}'\hat{i} + \dot{y}'\hat{j} + \dot{z}'\hat{k}$ is rate of change of $\underline{r}'$ as measered by an observer moving with S'. It may be called the rate of growing since in calculating it

we think of the vector $\underline{r}'$ as changing or growing, where as $\hat{i}, \hat{j}, \hat{k}$ remains constant. We write it as

$$\left(\frac{d\underline{r}'}{dt}\right)_{rotating} = \dot{x}'\hat{i} + \dot{y}'\hat{j} + \dot{z}'\hat{k}$$

2nd part of $\left(\frac{d\underline{r}'}{dt}\right)_{fixed}$ viz $\underline{\omega} \times \underline{r}'$ is due to rotation of triad $\hat{i}, \hat{j}, \hat{k}$ or rotation of axes and is not measured or felt by the rotating observer. It will be measured by an observer in fixed system and is called rate of transport. Thus for a vector $\underline{r}'$ in a rotating frame, the rate of change of $\underline{r}'$ observed w.r.t fixed system is:

$$\left(\frac{d\underline{r}'}{dt}\right)_{fixed} = \left(\frac{d\underline{r}'}{dt}\right)_{rotating} + \underline{\omega} \times \underline{r}' \longrightarrow (1)$$

This relation holds not only for the p.v but for any vector $\underline{f}$ associated with moving frame and we have

$$\left(\frac{d\underline{f}}{dt}\right)_{fixed} = \left(\frac{d\underline{f}}{dt}\right)_{rot} + \underline{\omega} \times \underline{f}$$

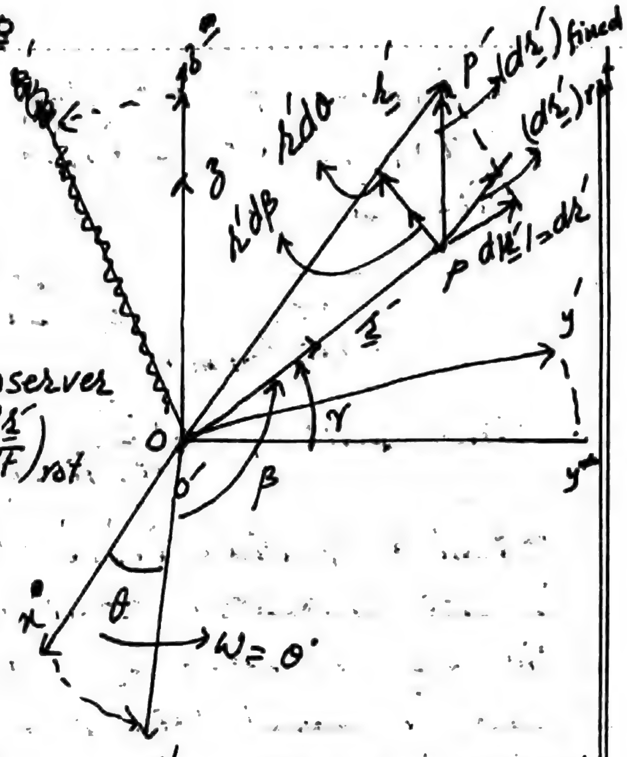
$\Rightarrow$ The operation of $\frac{d}{dt}$ in the fixed frame is transferred to

$$\left[\left(\frac{d}{dt}\right) + \underline{\omega} \times \cdot\,\right] \text{ in the rotating frame}$$

Physical Significance of Equation (1)

Suppose vector $\underline{r}' = \overrightarrow{o'P}$ swings to position $\overrightarrow{oP'}$ during infinitesimal interval of time dt. Then the observer in rotating frame measures two components

(a) dr' due to its change in magnitude
(b) $r'\,d\beta$ due to its rotation $d\beta$ relative to S' – system



~~Due~~ to the rotating observer, then, the derivative $\left(\frac{dr'}{dt}\right)_{rot}$ which the obeserver measures has components $\frac{dr'}{dt}$ and $r'\frac{d\beta}{dt} = r'\beta^{\circ}$

The remaining part of the total derivative not measured by the rotating observer has magnitude $r'\frac{d\theta}{dt}$ expressed as a vector is $\underline{\omega} \times \underline{r}'$. Thus

we have considered O & O' coincident for easiness and S' rotates about z'-axis with $\underline{\omega}$

$$\left(\frac{d\underline{r}'}{dt}\right)_{fixed} = \left(\frac{d\underline{r}'}{dt}\right)_{rot} + \underline{\omega} \times \underline{r}'$$

Corollary # If the rotating axes are fixed. in a rigid body i.e are body axes and O', P are two particles of rigid body rotating with angular velocity $\underline{\omega}$. Then

$$\left(\frac{d\underline{r}'}{dt}\right)_{rot} = \underline{0} \quad \text{because } \underline{r}' \text{ does not}$$

change both in magnitude (due to rigidity of body) and in direction w.r.t body axes because observer in body axes also rotate with same velocity $\underline{\omega}$ and their is no relative change of direction. Thus

$$\left(\frac{d\underline{r}'}{dt}\right)_{Fixed} = \underline{\omega} \times \underline{r}' \quad \text{for this case.}$$

Problem # Prove that the rotation angle of a rigid body displacement and also the instantaneous angular velocity vector, is independent of the choice of origin of the body system of axes. OR

Prove that the essence of rigid body constraint is that all particles of the body move and rotate together. OR

Prove that the angular velocity vector is same for all Co-ordinates systems in the rigid body

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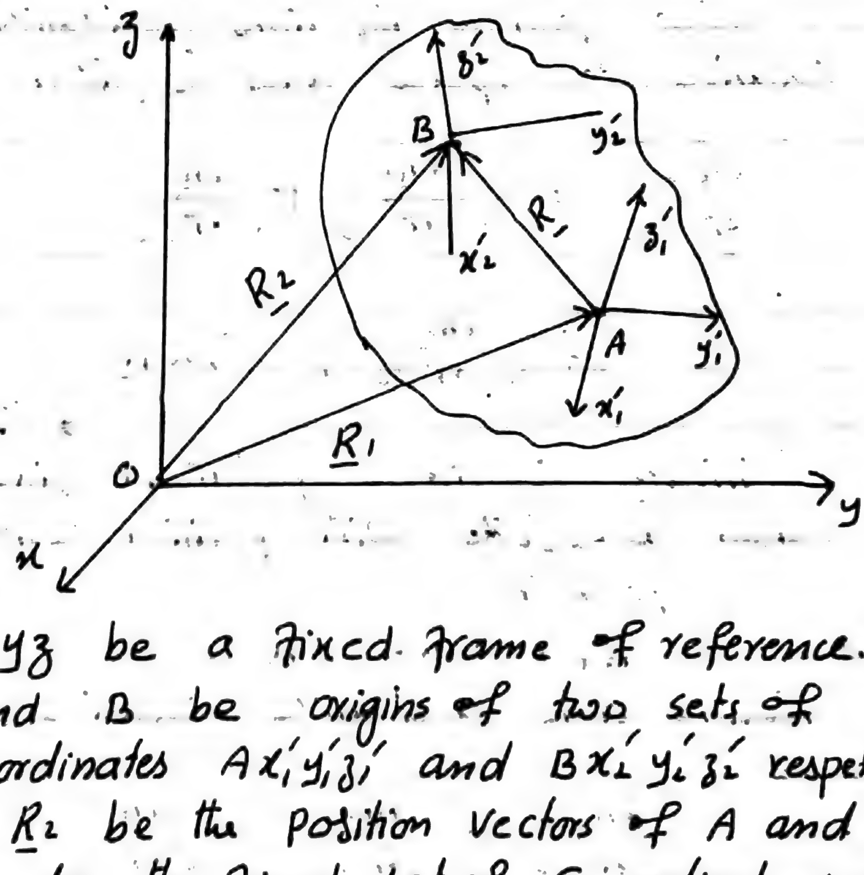
Proof #



Let $Oxyz$ be a fixed frame of reference. Let A and B be origins of two sets of body co-ordinates $Ax_1'y_1'z_1'$ and $Bx_2'y_2'z_2'$ respectively and $\underline{R_1}$, $\underline{R_2}$ be the position vectors of A and B relative to the fixed set of co-ordinate axes. The difference vector $\underline{R}$ is given by

$$\underline{R} = \underline{R_2} - \underline{R_1}$$

$$\underline{R_2} = \underline{R_1} + \underline{R} \qquad \to (2)$$

Considering B a point defined in system with origin at A. We have

$$\left(\frac{d\underline{R}}{dt}\right)_s = \left(\frac{d\underline{R}}{dt}\right)_b + \underline{\omega}_1 \times \underline{R}$$

$\because$ body is rigid

$\therefore$ $\underline{R}$ will be constant as seen in body axes frame with origin at A

$$\Rightarrow \quad \left(\frac{d\underline{R}}{dt}\right)_b = \underline{0}$$

$$\Rightarrow \quad \left(\frac{d\underline{R}}{dt}\right)_s = \underline{\omega}_1 \times \underline{R} \longrightarrow ②$$

where ω_1 is velocity of body about A

Now from ① the time derivative of $\underline{R}_2$ relative to space axes is given by

$$\left(\frac{d\underline{R}_2}{dt}\right)_s = \left(\frac{d\underline{R}_1}{dt}\right)_s + \left(\frac{d\underline{R}}{dt}\right)_s$$

$$= \left(\frac{d\underline{R}_1}{dt}\right)_s + \underline{\omega}_1 \times \underline{R} \quad \text{by ②} \longrightarrow ③$$

Similarly consider point A fixed in the system $Bx_2' y_2' z_2'$ with position vector $-\underline{R}$

from ①

$$\underline{R}_1 = \underline{R}_2 - \underline{R}$$

Taking time derivative relative to space axes

$$\left(\frac{d\underline{R}_1}{dt}\right)_s = \left(\frac{d\underline{R}_2}{dt}\right)_s - \left(\frac{d\underline{R}}{dt}\right)_s$$

$$= \left(\frac{d\underline{R}_2}{dt}\right)_s - \underline{\omega}_2 \times \underline{R}$$

where $\underline{\omega}_2$ is angular velocity of body about B

$$\Rightarrow \left(\frac{d\underline{R}_2}{dt}\right)_s = \left(\frac{d\underline{R}_1}{dt}\right)_s + \underline{\omega}_2 \times \underline{R} \longrightarrow ④$$

By (3) & (4), we get

$$\underline{\omega}_1 \times \underline{R} = \underline{\omega}_2 \times \underline{R}$$

$$\Rightarrow \underline{\omega}_1 = \underline{\omega}_2$$

$\Rightarrow$ The angular velocity vector is same for all Co-ordinates system in the rigid body.

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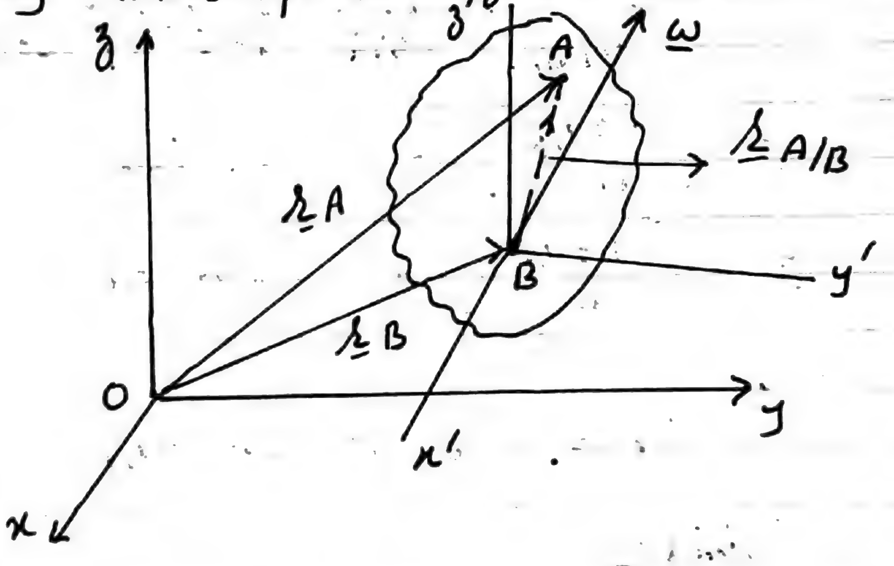
General Motion of a Rigid Body*

In general case where no point of the rigid body is fixed but body translate as well rotate we may regard the body's motion as the displacement of one point of it to a new position O′ (say) followed by an appropriate turning about a certain axis through O′ until the body takes up the required orientation.



Consider a rigid body which has an angular velocity $\underline{\omega}$. At any time t during the displacement of body we choose any convenient point B of body and attach with it a body set of axis $Bx'y'z'$ which rotates with velocity $\underline{\omega}$. Now we may consider that

point B translate and body rotates about B instantaneously with velocity $\underline{\omega}$. If A is any other point of body with p.v $\underline{r}_{A/B}$ relative to B. Then

$$\underline{r}_A = \underline{r}_B + \underline{r}_{A/B}$$

and time derivative w.r.t space axes is

$$\left(\frac{d\underline{r}_A}{dt}\right)_s = \left(\frac{d\underline{r}_B}{dt}\right)_s + \left(\frac{d\underline{r}_{A/B}}{dt}\right)_s$$

$$\underline{V}_A = \underline{V}_B + \left(\frac{d\underline{r}_{A/B}}{dt}\right)_s$$

But $\left(\frac{d\underline{r}_{A/B}}{dt}\right)_s = \left(\frac{d\underline{r}_{A/B}}{dt}\right)_b + \underline{\omega} \times \underline{r}_{A/B}$

$\because$ Body is ridigid

$\therefore$ $\underline{r}_{A/B}$ remains constant w.r.t to an observer in body axes rotating with velocity of body

$$\Rightarrow \left(\frac{d\underline{r}_{A/B}}{dt}\right)_b = \underline{0} = \underline{V}_{rel}$$

$$\Rightarrow \left(\frac{d\underline{r}_{A/B}}{dt}\right)_s = \underline{\omega} \times \underline{r}_{A/B}$$

Thus

$$\underline{V}_A = \underline{V}_B + \underline{\omega} \times \underline{r}_{A/B}$$

Similarly

$$\underline{a}_A = \underline{a}_B + \dot{\underline{\omega}} \times \underline{r}_{A/B} + \underline{\omega} \times (\underline{\omega} \times \underline{r}_{A/B})$$

The selection of the reference point B is quite arbitrary. In practice point B is chosen for convenience as some point on the body whose motion is known in whole or in part

The reference point B is also called base point.

If A is selected as the reference point, the relative equations become.

$$\underline{V}_B = \underline{V}_A + \underline{\omega} \times \underline{r}_{B/A}$$

$$\underline{a}_B = \underline{a}_A + \underline{\dot{\omega}} \times \underline{r}_{B/A} + \underline{\omega} \times (\underline{\omega} \times \underline{r}_{B/A})$$

where $\underline{r}_{B/A} = -\underline{r}_{A/B}$.

It should be clear that $\underline{\omega}$ and hence $\underline{\dot{\omega}}$ are the same vectors for both cases because the absolute angular velocity $\underline{\omega}$ is independent of the choice of reference point.

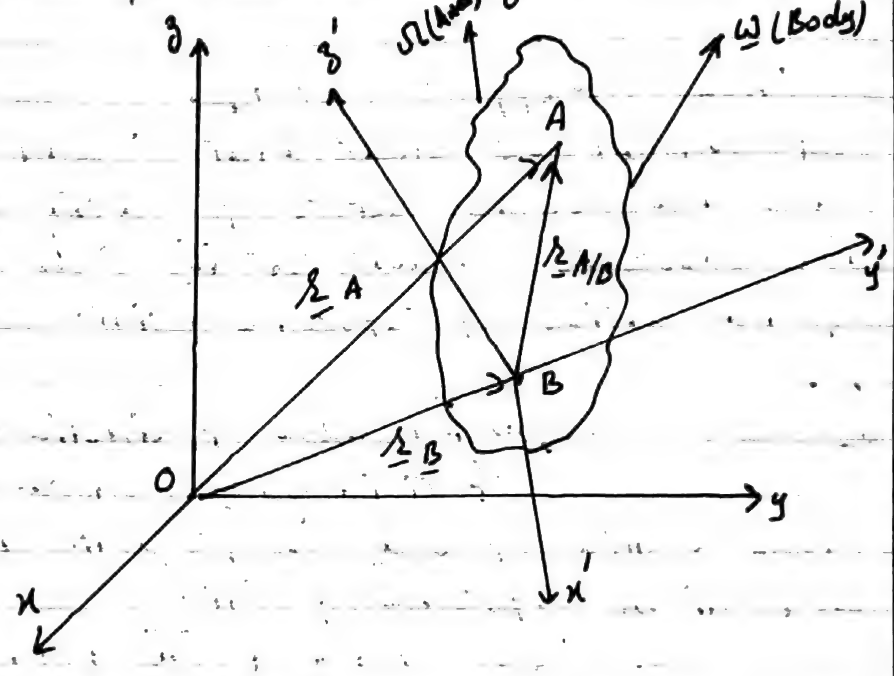
Reremarks #(1) In case of kinetic equations of rigid body general motion the mass centre of body is frequently the most convenient reference point.

(2) If the points A & B selected above represent the ends of a rigid control link in a special mechanism where the end connections act as ball and socket joints, it is necessary to impose certain kinematics requirements. Clearly any rotation of the link about its own axis AB does not effect the action of the Link. Thus the angular velocity $\underline{\omega}_n$, whose vector is normal to the link describes its action. It is necessary, therefore, that $\underline{\omega}_n$ & $\underline{r}_{A/B}$ be at right angles and this condition is satisfied if $\underline{\omega}_n \cdot \underline{r}_{A/B} = 0$. Similarly, it is only component α_n of the angular acc. of the link normal to AB which affects its action, so that $\underline{\alpha}_n \cdot \underline{r}_{A/B} = 0$

must also hold.

More General Motion of Rigid Body

A more general motion of rigid body in space we use reference axes which translate with the rigid body as well as rotate with an absolute angular velocity which may be different from the absolute angular velocity $\underline{\omega}$



Consider the general motion of a rigid body which translates as well rotates with an angular velocity $\underline{\omega}$. At any instant during motion. Consider point B as reference point of body and reference axes $Bx'y'z'$ with origin attached to point B but which rotate with an angular velocity Ω different from ω. Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors attached to x'-y'-z'. If A is any other point of rigid body, Then

$$\underline{v}_A = \underline{v}_B + \underline{\Omega} \times \underline{r}_{A/B} + \underline{v}_{rel}$$

$$\underline{a}_A = \underline{a}_B + \dot{\underline{\Omega}} \times \underline{r}_{A/B} + \underline{\Omega} \times (\underline{\Omega} \times \underline{r}_{A/B}) + 2\underline{\Omega} \times \underline{v}_{rel} + \underline{a}_{rel}$$

where $\underline{v}_{rel} = \dot{x}'\hat{i} + \dot{y}'\hat{j} + \dot{z}'\hat{k}$ and $\underline{a}_{rel} = \ddot{x}'\hat{i} + \ddot{y}'\hat{j} + \ddot{z}'\hat{k}$ are velocity and acceleration of point A relative to $x'-y'-z'$ measured by an obeserver attached to $x'-y'-z'$

We note that $\underline{r}_{A/B}$ remains Constant in magnitude. For points A & B fixed to rigid body but it will change direction w.r.t $x'-y'-z'$ when the angular velocity $\underline{\Omega}$ of the axes is different from the angular velocity $\underline{\omega}$ of the body. Also if $x'-y'-z'$ are rigidly attached to body (i.e. are body axes), then $\underline{\Omega} = \underline{\omega}$ and $\underline{v}_{rel}$ and $\underline{a}_{rel}$ are both zero

Transformation of time derivative between the two systems of axes is for any vector $\underline{f}$ is given by

$$\left(\frac{d\underline{f}}{dt}\right)_{fixed} = \left(\frac{d\underline{f}}{dt}\right)_{rot} + \underline{\Omega} \times \underline{f}$$

$$\rightarrow \left(\frac{d[\ \ ]}{dt}\right)_{fixed} = \left(\frac{d[\ \ ]}{dt}\right)_{rot} + \underline{\Omega} \times [\ \ ]$$

where [ ] stands for any vector $\underline{f}$ expressible both in fixed axes and rotating axes

If we apply the operator to itself, we get 2nd order operator as

$$\left(\frac{d^2[\ \ ]}{dt^2}\right)_{fixed} = \left(\frac{d^2[\ \ ]}{dt^2}\right)_{rot} + \dot{\underline{\Omega}} \times [\ \ ] + \underline{\Omega} \times (\underline{\Omega} \times [\ \ ]) + 2\underline{\Omega} \times \left(\frac{d[\ \ ]}{dt}\right)_{rot}$$

CHASLES' THEOREM

Statement

The most general displacement of a rigid body is a translation and a rotation about some base point.

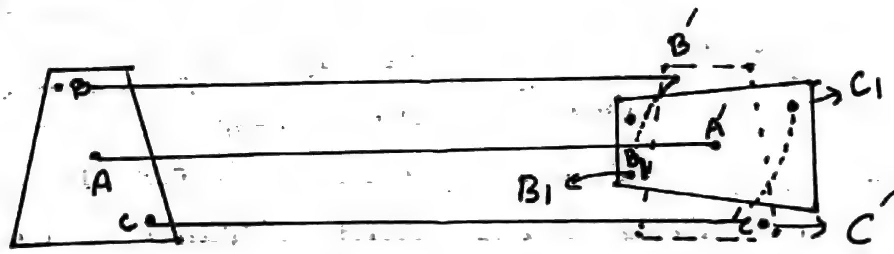
OR

It states that an arbitrary displacement of a rigid body consists of translation of the fixed point and rotation about an axis passing through the fixed.

OR

The most general displacement of a rigid body is a translation plus a rotation

Proof



Consider a general displacement of rigid body in which the body is not restricted to turn about a fixed point. Let A, B, C be the initial positions of the three non-colinear particles of the body. Let A', B_1, C_1 be the final positions of these particles. The we note at once that this displacement is carried out in two steps.

(1) A translation which takes A to A', B to B', C to C' where

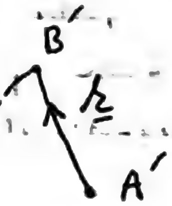
$\overrightarrow{BB'}$

(2) A rotation about A' which takes B' to B_1 and C' to C_1

In the above description we have considered rotation of body about a particle (pt of body) originally at A. Such a particle (point) is called base point. The base point can be changed. With the change of base point, the translation involved will be changed but rotation is independent of the base point. For the motion to be general angular displacents must be independent of the base point and rotation must also be independent of the base point. We prove these two condition.

For base point A' #

Let $\underline{r}_{A'}$ be p.v of A' relative to some fixed axes. Let $\underline{\omega}_1$ be angular velocity of the body which takes B' to B_1 and C_1' to C_1 when body rotates about some axis through A'. For time dt. Let $\underline{r}_{B_1'}$ be p.v of B_1 relative to fixed axis. and $\underline{r}_{B_1/A} = \underline{r}$



Displacement of B' during time interval dt is given by

$$d\underline{r} = \underline{V}_{A}'dt + \underline{\omega}_1 dt \times \underline{r} \longrightarrow (9)$$

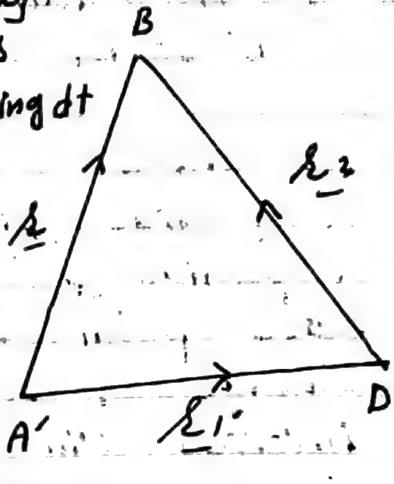
$$\frac{d\underline{r}}{dt} = \underline{V}_{A'} + \underline{\omega}_1 \times \underline{r}$$

$$V_{B'} = \underline{V}_{A'} + \underline{\omega}_1 \times \underline{r} \longrightarrow ①$$

Now consider another point D as base point such that

$$\underline{r}_{D/A'} = \underline{r}_1 \quad \& \quad \underline{r}_{B'/D} = \underline{r}_2$$

and $\underline{\omega}_2$ be angular velocity of body about D, which takes B' to B_1 and C' to C_1 during dt. Then change in linear displacement $d\underline{r}_2$ is



$$d\underline{r}_2 = \underline{V}_D dt + \underline{\omega}_2 dt \times \underline{r}_2 \rightarrow ②$$

Also

$$\underline{V}_D = V_{A'} + \underline{\omega}_1 \times \underline{r}_1 \rightarrow ③$$

Considering A' a point in rotating frame at D, we have

$$\underline{V}_{A'} = \underline{V}_D - \underline{\omega}_2 \times \underline{r}_1$$

$$\underline{V}_D = \underline{V}_{A'} + \underline{\omega}_2 \times \underline{r}_1 \rightarrow ④$$

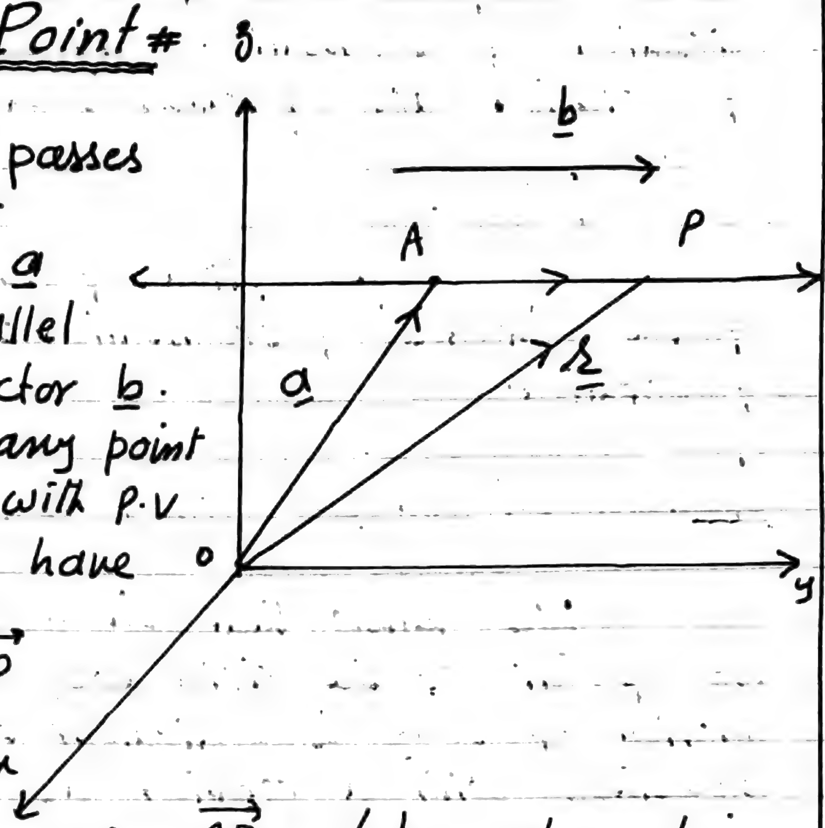
By ③ and ④, we have

$$\underline{\omega}_1 = \underline{\omega}_2$$

$\Rightarrow$ rotation is independent of base point. But from ⓐ and ② we note that by change of base point the translation involved is changed. Hence proved.

Remarks# Chasles also proved a stronger form of the theorem which states that it is possible to choose origin of the body set of axes so that the translation is in the same direction as the axis of rotation. Such a combination of translation and rotation is called a screw motion. This version of the theorem is of little use now-a-days.

Vector Equation of a Line Passing through a given Point



Let a line passes through point A with p.v $\underline{a}$ and is parallel to given vector $\underline{b}$. If P is any point on the line with p.v $\underline{r}$. Then we have

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

$\because$ $\overrightarrow{AP}$ is ||al to vector $\underline{b}$ $\therefore$ $\overrightarrow{AP} = t\,\underline{b}$ where t is scalar

$$\overrightarrow{OP} = \overrightarrow{OA} + t\,\underline{b}$$

$\underline{r} = \underline{a} + t\,\underline{b}$ Required equation of line.

Screw Motion

Problem* # (a) Define screw motion and pitch of the screw

(b) Prove that the general rigid body motion is a screw motion

Sol # (a) **Definition #.**

Such a displacement of rigid body in which translation is in the same direction as the direction of axis of

rotation, is called screw motion.

OR

The displacement of rigid body in which direction of translation and direction of axis of rotation are parallel, is called screw motion.

Pitch

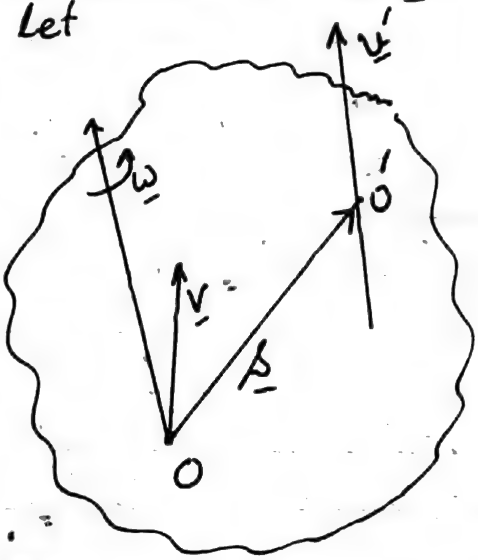
The pitch of screw is the ratio of the distance of translation to the angle of rotation.

General Rigid Body Motion as Screw Motion

We know that in general case where no point of the rigid body is fixed, any motion of body is equivalent to a translation together with a rotation about an axis.

We now show that position of axis of rotation can be so chosen that the translation is parallel to it.

Let the point O, fixed in the body, have velocity $\underline{v}$ and let $\underline{\omega}$ be the instantaneous angular velocity of the body. If O′ is any other point of body with $\overrightarrow{OO'} = \underline{s}$ and $\underline{v}'$ be the velocity of O′, then we have



$$\underline{V}_{O'} = \underline{V}_O + \underline{\omega} \times \underline{s}.$$

$$\underline{V}' = \underline{V} + \underline{\omega} \times \underline{s} \longrightarrow (1)$$

Now if we choose point O′ such that

$\underline{V}'$ is parallel to $\underline{\omega}$ i.e parallel to the axis of rotation, then

$$\underline{\omega} \times \underline{V}' = \underline{0}$$

and from ①, we have

$$\underline{0} = \underline{\omega} \times \underline{V} + \underline{\omega} \times (\underline{\omega} \times \underline{s})$$

$$= \underline{\omega} \times \underline{V} + (\underline{\omega} \cdot \underline{s})\,\underline{\omega} - \omega^2 \underline{s}$$

If $\underline{\omega} \neq \underline{0}$, then we have

$$\underline{s} = \frac{\underline{\omega} \times \underline{V}}{\omega^2} + \frac{(\underline{\omega} \cdot \underline{s})\,\underline{\omega}}{\omega^2}$$

$$= \frac{\underline{\omega} \times \underline{V}}{\omega^2} + \lambda\,\underline{\omega} \longrightarrow ②$$

where $\lambda = \dfrac{\underline{\omega} \cdot \underline{s}}{\omega^2}$

Equation ② is equation line through point with p.v $\dfrac{\underline{\omega} \times \underline{V}}{\omega^2}$ and parallel to $\underline{\omega}$.

$\because$ $\underline{s}$ is p.v of point O' and ② is equation of line

$\therefore$ When λ varies p.v of O' traces out a straight line parallel to $\underline{\omega}$ i.e O' moves on a straight line parallel to the axis of rotation. Hence the instantaneous motion of the body is a screw motion about this line. This line is called the axis of the screw or central axis. Every point on the axis moves along it and the body turns about the axis.

Now the velocity $\underline{V}$ at any point P of the body where $\overrightarrow{OP} = \underline{r}$ is given by

$$\underline{V} = \underline{v} + \underline{\omega} \times \underline{r}$$

Hence
$$\underline{V}\cdot\underline{\omega} = \underline{v}\cdot\underline{\omega} + (\underline{\omega}\times\underline{r})\cdot\underline{\omega}$$
$$= \underline{v}\cdot\underline{\omega} + 0$$

$$\underline{V}\cdot\underline{\omega} = \underline{v}\cdot\underline{\omega}$$
$\Rightarrow$ the scalar quantity $\underline{v}\cdot\underline{\omega}$ is invariant for the system at any instant. Also ω^2 is invariant. We write

$$p = \frac{\underline{v}\cdot\underline{\omega}}{\omega^2}$$
p is then, another invariant.

For any point on the axis of the screw, the velocity $\underline{v}'$ of the point is parallel to $\underline{\omega}$. Therefor

$$p = \frac{\underline{v}'\cdot\underline{\omega}}{\omega^2} = \frac{v'\omega\cos 0^\circ}{\omega^2}$$
$$= \frac{v'}{\omega} = \frac{\left(\frac{\text{Distance along axis}}{\text{time}}\right)}{\left(\frac{\text{Angle of turn}}{\text{time}}\right)}.$$

$$p = \frac{\text{Distance along axis of screw}}{\text{Angle of turn}}$$

$\Rightarrow$ p is the pitch of the screw.

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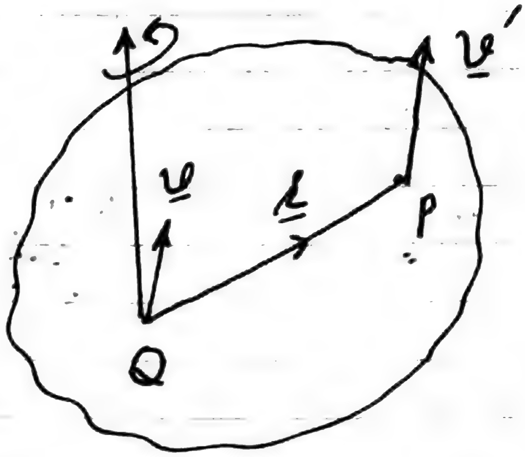
Problem # A rigid has spin $\underline{\omega}$ and particle Q of the body has vel $\underline{v}$. Show that every particle P of the body with velocity vector parallel to $\underline{\omega}$ lies the line with vector equation

$$\underline{r} = \overrightarrow{QP} = \frac{(\underline{\omega}\times\underline{V})}{\omega^2} + \mu\,\underline{\omega}$$
where $\mu = \dfrac{\underline{\omega}\cdot\underline{r}}{\omega^2}$

Sol # Let velocity of P be $\underline{V}'$ which is parallel to $\underline{\omega}$.

Then



$$\underline{V}' = \underline{V} + \underline{\omega} \times \vec{QP}$$

$$= \underline{V} + \underline{\omega} \times \underline{r}$$

$$\vec{QP} = \underline{r}$$

$\because$ $\underline{V}'$ is llal to $\underline{\omega}$

$$\therefore \quad \underline{\omega} \times \underline{V}' = \underline{\omega} \times \underline{V} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

$$\underline{0} = \underline{\omega} \times \underline{V} + (\underline{\omega} \cdot \underline{r})\underline{\omega} - (\underline{\omega} \cdot \underline{\omega})\underline{r}$$

$$= \underline{\omega} \times \underline{V} + (\underline{\omega} \cdot \underline{r})\underline{\omega} - \omega^2 \underline{r}$$

$$\underline{r} = \frac{\underline{\omega} \times \underline{V}}{\omega^2} + \left(\frac{\underline{\omega} \cdot \underline{r}}{\omega^2}\right)\underline{\omega}$$

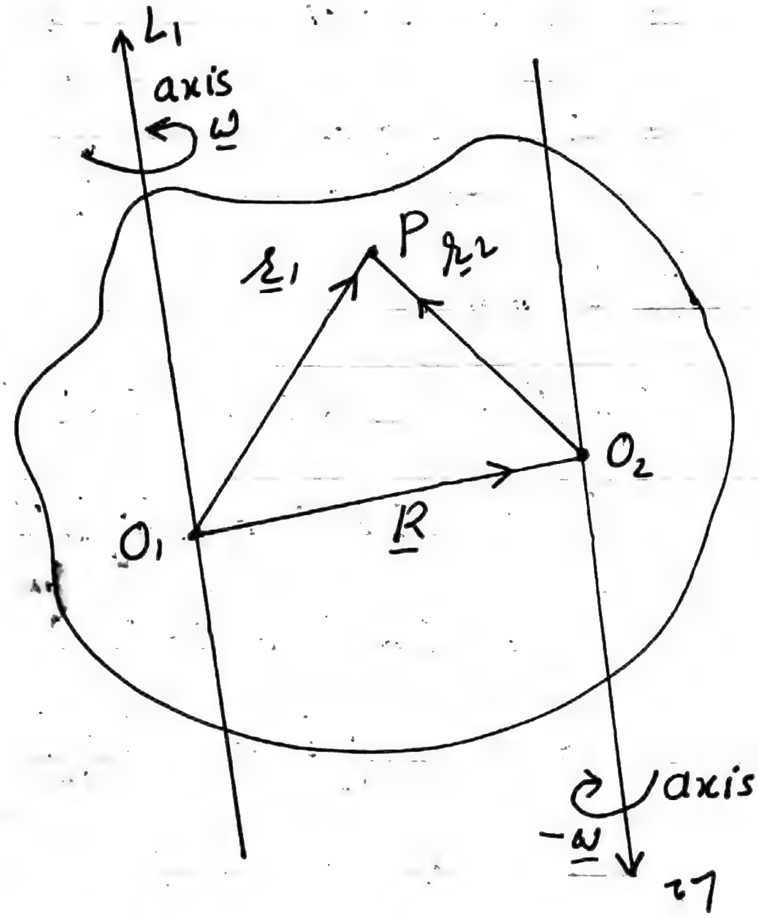
$$r = \underline{a} + \mu \underline{\omega}$$

where $\underline{a} = \frac{1}{\omega^2}(\underline{\omega} \times \underline{V})$ $\quad \mu = \frac{\underline{\omega} \cdot \underline{r}}{\omega^2}$
which is an equation of line passing through a poin with p.v. $\underline{a} = \frac{\underline{\omega} \times \underline{V}}{\omega^2}$ and parallel to $\underline{\omega}$. This show that any particle P of the body with velocity parallel to $\underline{\omega}$ lies on the line

$$\underline{r} = \frac{\underline{\omega} \times \underline{V}}{\omega^2} + \mu \underline{\omega}.$$

Theorem*# Show that two equal and opposite rotations of a rigid body about two distinit parallel axes are equivalent to a translation of the body #

Proof



Let L_1 and L_2 be two distinct parallel axes of rotation of the rigid body. Since the rotations of rigid body about L_1 and L_2 are equal and opposite, therefore if $\underline{\omega}$ is angular velocity about L_1 anti-clockwise, then $-\underline{\omega}$ will be angular velocity about L_2, clockwise.

Let O_1 and O_2 be points on the axis of rotations L_1 & L_2 such that $\overrightarrow{O_1O_2} = \underline{R}$. Let P be any point of rigid body such that

$$\overrightarrow{O_1P} = \underline{r}_1 \quad \& \quad \overrightarrow{O_2P} = \underline{r}_2$$

The velocity of P is

$$\underline{V}_P = (\text{vel of P due to rot about } L_1) + (\text{Vel of P due to rot. about } L_2)$$

$$= (+\underline{\omega}) \times \underline{r}_1 + (-\underline{\omega}) \times \underline{r}_2$$

$$= \underline{\omega} \times (\underline{r}_1 - \underline{r}_2)$$

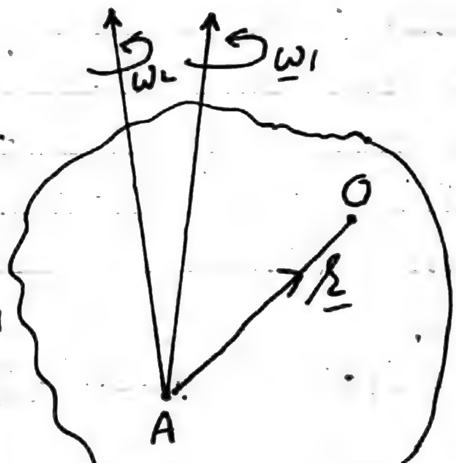
$\because \underline{r}_1 = \underline{R} + \underline{r}_2$

$$\underline{V}_P = \underline{\omega} \times \underline{R}$$

$\therefore \underline{r}_1 - \underline{r}_2 = \underline{R}$

which gives only linear velocity of point P and is independent of position of P. Thus every point of rigid body moves instantaneously with this velocity i.e the rigid body has a translation because during translation of a rigid body each particle moves with same velocity.

<u>Theorem</u> # If a rigid body moves in such a way that a velocity $\underline{V}_1$ of a point O fixed in it together with a spin $\underline{\omega}_1$ is to be combined with a velocity $\underline{V}_2$ of O and a spin $\underline{\omega}_2$ of the body, then the net motion is a velocity $\underline{V}_1 + \underline{V}_2$ of O and a spin $\underline{\omega}_1 + \underline{\omega}_2$ of the body



<u>Proof</u> # Consider a rigid body spinning about a point A of it and $\underline{\omega}_1$, $\underline{\omega}_2$ be its instantaneous angular velocities about some axes through A. Let O be

point fixed in body such that

$$\vec{AO} = \underline{r}$$

Velocity $\underline{v}_1$ of O due to spin $\underline{\omega}_1$ is given by

$$\underline{v}_1 = \underline{\omega}_1 \times \underline{r}$$

Velocity $\underline{v}_2$ due to spin $\underline{\omega}_2$ is

$$\underline{v}_2 = \underline{\omega}_2 \times \underline{r}$$

Net velocity of O due to both spins is

$$\underline{V} = \underline{v}_1 + \underline{v}_2$$

$$= \underline{\omega}_1 \times \underline{r} + \underline{\omega}_2 \times \underline{r}$$

$$\underline{v}_1 + \underline{v}_2 = (\underline{\omega}_1 + \underline{\omega}_2) \times \underline{r}$$

Thus the net motion is a velocity $\underline{v}_1 + \underline{v}_2$ of O and a spin $\underline{\omega}_1 + \underline{\omega}_2$ of the body.

Rolling Wheel

Problem # (a) Define rolling between Two bodies

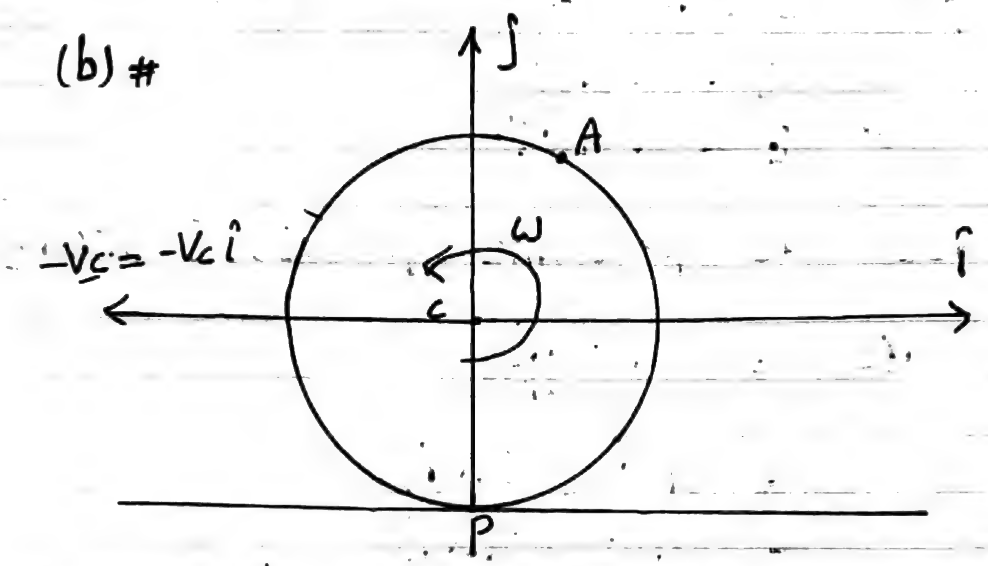
(b) A circular disc of radius 'a' is rolling without slippling along a fixed straight line. If its angular velocity is constant, the find velocity and acceleration of its centre and velocity and acceleration of particle on its rim not in contact with the line on which disc rolls

Sol# Rolling #

If two bodies have point

or line contact, then rolling is said to occur between them if the velocities of contiguous points are equal. e.g if two rigid bodies are in contact at some point P, then velocity of P considered as a point of one body is equal to the velocity of P considered as the point of other body.

(b) #



Suppose the disc is rolling towards left with angular velocity $\underline{\omega}$. Taking a co-ordinate system with origin at centre c of disc and $\underline{V_c}$ as velocity of centre. Let P be the point of contact. Then velocity p considered as a point of the disc is equal to the velocity of p considered as a point of line.

The velocity of P as a point of disc is

$$\underline{V_P} = \underline{V_c} + \underline{\omega} \times \overrightarrow{CP}$$

Since the line is fixed, the P as a point of line is fixed. $\underline{V_P} = 0$

$$\Rightarrow \quad 0 = \underline{V_c} + \underline{\omega} \times \overrightarrow{CP}$$

Now $\underline{V_c} = -V_c\hat{i}$, $\quad \overrightarrow{CP} = -a\hat{j}$,

$\underline{\omega} = \omega = \omega\hat{k}$

$$\underline{0} = -V_c\hat{i} + \omega\hat{k}\times(-a\hat{j})$$

$$= -V_c\hat{i} - a\omega(\hat{k}\times\hat{j})$$

$$\underline{0} = -V_c\hat{i} - a\omega(-\hat{i})$$

$$\Rightarrow (V_c + a\omega)\hat{i} = \underline{0}$$

$$\Rightarrow V_c + a\omega = 0 \qquad \hat{i} \neq \underline{0}$$

$$V_c = -a\omega$$

or $\underline{V_c} = -a\omega\hat{i}$

Acc $\underline{a_c}$ of centre of disc is given by

$$\underline{a_c} = -a\omega\frac{d\hat{i}}{dt}$$

$$= -a\omega(\underline{\omega}\times\hat{i})$$

$$= -a\omega(\omega\hat{k}\times\hat{i})$$

$$= -a\omega^2\hat{j}$$

$$= a\omega^2(-\hat{j})$$

$\Rightarrow$ magnitude of the acc $= a\omega^2$

Let A be any particle on the rim of the disc. Then velocity of A is given by

$$\underline{a}_A = \underline{V_c} + \overset{\underline{\omega}\times}{}(\underline{\omega}\times\overrightarrow{CA})$$

$$= \underline{V_c} + \underline{\omega}\times(\underline{\omega}\times\overrightarrow{CA})$$

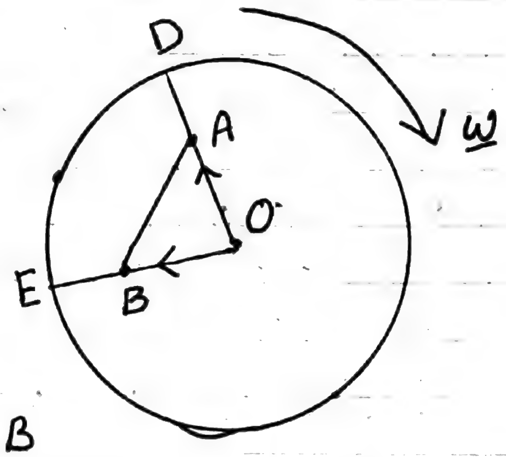
$$= -a\omega^2\hat{j} + (\underline{\omega}\cdot\overrightarrow{CA})\underline{\omega} - (\underline{\omega}\cdot\underline{\omega})\overrightarrow{CA}$$

$$= -a\omega^2\hat{j} + \underline{0} - \omega^2\overrightarrow{CA}$$

$$\underline{a}_A = -(a\omega^2\hat{j} + \omega^2\overrightarrow{CA})$$

Problem# A wheel is rolling without slipping along a plane without slipping and with angular velocity $\underline{\omega}$. A and B are taken at different distances from the centre on different spokes. Show that at any time t the vel of A relative B is $\underline{\omega} \times \overrightarrow{AB}$ and that of B relative to A is $\underline{\omega} \times \overrightarrow{BA}$. What will be their relative velocity if both A & B lie on the same spoke and at different distances from the centre.

Sol#



Let O be the centre of wheel and $\underline{V}$ be the velocity of O relative to some fixed axes. Let A & B be points on spokes OD and OE.

Let $\underline{V}_A$, $\underline{V}_B$ be velocities of A and B relative fixed axes. Then

$$\underline{V}_A = \underline{V} + \underline{\omega} \times \overrightarrow{OA} \quad \rightarrow ①$$

$$\underline{V}_B = \underline{V} + \underline{\omega} \times \overrightarrow{OB} \quad \longrightarrow ②$$

The velocity of B relative to A is

$$\underline{V}_{B/A} = \underline{V}_{B/A} = \underline{V}_B - \underline{V}_A = \underline{\omega} \times (\overrightarrow{OB} - \overrightarrow{OA})$$

$$= \underline{\omega} \times \overrightarrow{AB}$$

Velocity of A relative to B is

$$\underline{V}_{A/B} = \underline{V}_A - \underline{V}_B = \underline{\omega} \times (\overrightarrow{OA} - \overrightarrow{OB}) = \underline{\omega} \times \overrightarrow{BA}$$

If both points A and lie on the same spoke, $\overrightarrow{AB}$ and $\overrightarrow{BA}$ lie on the same spoke and relative velocities will be

$$\underline{V}_{A/B} = \underline{\omega} \times \overrightarrow{BA}$$

$$\underline{V}_{B/A} = \underline{\omega} \times \overrightarrow{AB}$$

Problem*# A rigid bod S has a spin $\underline{\omega}$ and particle A of S has velocity $\underline{V}$. Show that every particle P of S with velocity vector parallel to ω lies on the line $\overrightarrow{AP} \equiv (\underline{\omega} \times \underline{V})\omega^2 + \mu \underline{\omega}$, μ being an arbitrary scalar.

The instantaneous velocities of particles at points $(a,0,0)$, $(0, a/\sqrt{3}, 0)$, $(0,0,2a)$ of rigid body are $[u,0,0]$, $[u, 0, u]$ $[u+v, -2v\sqrt{3}, v/2]$ respectively, to rectangular Cartesian frame. Find the magnitude and direction of the spin of the body and the point at which central axis cuts the xz-plane.

Sol# Let A, B, C be points $(a,0,0)$, $(0, \frac{a}{\sqrt{3}}, 0)$, $(0,0,2a)$

and let $\underline{\omega} = [\omega_1, \omega_2, \omega_3]$

Let $\underline{r}_A$, $\underline{r}_B$, $\underline{r}_C$ be position vectors of points A, B, C relative to fixed Cartesian frame. Then

$$\underline{r}_A = \overrightarrow{OA} = [a;0;0] = a\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\underline{r}_B = \overrightarrow{OB} = [0, \tfrac{a}{\sqrt{3}}, 0] = 0\hat{i} + \tfrac{a}{\sqrt{3}}\hat{j} + 0\hat{k}$$

$\underline{r}_c = [0, 0, 2a] = 0\hat{i} + 0\hat{j} + 2a\hat{k}$

Since velocity of A is given, therefore we take this point body as fixed point or base point

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = [0, \frac{a}{\sqrt{3}}, 0] - [a, 0, 0]$

$\underline{r}_1 = [-a, \frac{a}{\sqrt{3}}, 0]$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -[a, 0, 0] + [0, 0, 2a]$

$= [-a, 0, 2a]$

Let $\underline{v}_1$, $\underline{v}_2$ be velocities of B and C as relative to reference point A as seen by an observer in fixed frame.

$\underline{V}_1 = \underline{\omega} \times \underline{r}_1 \rightarrow ①$

But $\underline{V}_1 = \underline{V}_{B/A} = \underline{V}_B - V_A$

$= [U, 0, v] - [U, 0, 0]$

$= [0, 0, v]$

using in ①

$$[0, 0, v] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ -a & a/\sqrt{3} & 0 \end{vmatrix}$$

$$= \hat{i}\left(-\frac{a}{\sqrt{3}}\omega_3\right) - \hat{j}(a\omega_3) + \hat{k}\left(\frac{a\omega_1}{\sqrt{3}} + a\omega_2\right)$$

$\Rightarrow \omega_3 = 0 \qquad \frac{a\omega_1}{\sqrt{3}} + a\omega_2 = v$

$$a\omega_1 + a\omega_2\sqrt{3} = \sqrt{3}\,v \rightarrow (2)$$

Again $\underline{v}_2 = \underline{V}_{C/A} = \underline{V}_C - \underline{V}_A$

$$= [u+v, -v\sqrt{3}, \tfrac{v}{2}] - [u, 0, 0]$$

$$= [v, -v\sqrt{3}, v/2]$$

But

$$\underline{v}_2 = \underline{\omega} \times \underline{r}_2$$

$$\Rightarrow [v, -v\sqrt{3}, \tfrac{v}{2}] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ -a & 0 & 2a \end{vmatrix}$$

$$= \hat{i}(2a\omega_2) - \hat{j}(2a\omega_1 + a\omega_3) + \hat{k}(a\omega_2)$$

$$\Rightarrow \quad a\omega_2 = \frac{v}{2} \rightarrow (3)$$

$$2a\omega_1 + a\omega_3 = v\sqrt{3} \rightarrow (4)$$

putting $\omega_3 = 0$ in (4).

$\omega_1 = \dfrac{\sqrt{3}\,v}{2a}$ from (3) $\omega_2 = \dfrac{v}{2a}$

So spin of body is

$$\underline{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$$

$$= \frac{\sqrt{3}\,v}{2a}\hat{i} + \frac{v}{2a}\hat{j} + 0\hat{k}$$

$$= \frac{v}{2a}[\sqrt{3}, 1, 0]$$

$$|\underline{\omega}| = \omega = \frac{v}{a}$$

Direction of the Spin

spin of the body is

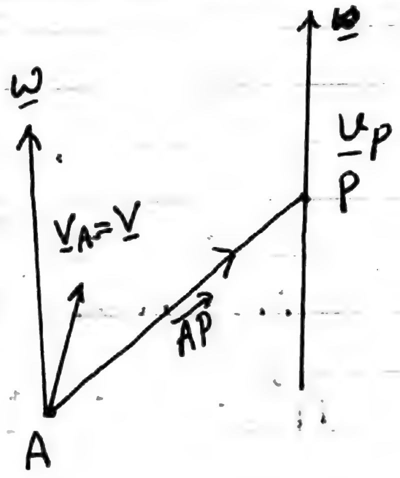
$$\underline{\omega} = \frac{\sqrt{3}\,u}{2a}\hat{i} + \frac{u}{2a}\hat{j}$$

Suppose that $\underline{\omega}$ (and hence instantaneous axis of rotation) makes angle θ with x-axis, then

$$\tan\theta = \frac{\left(\frac{u}{2a}\right)}{\left(\frac{\sqrt{3}\,u}{2a}\right)} = \frac{1}{\sqrt{3}}$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$$

Equation Central axis and its Intersection with xz-plane



Let P be any other point of body such that velocity $\underline{V}_P$ of P is parallel to $\underline{\omega}$ i.e P lies on the central axis. Then

$$\underline{V}_P = \underline{V}_A + \underline{\omega} \times \overrightarrow{AP}$$

$\because$ $\underline{\omega}$ and $\underline{V}_P$ are parallel

$$\underline{\omega} \times \underline{V}_P = \underline{\omega} \times \underline{V}_A + \underline{\omega} \times (\underline{\omega} \times \overrightarrow{AP})$$

$$0 = \underline{\omega} \times \underline{V} + (\underline{\omega}\cdot\overrightarrow{AP})\underline{\omega} - (\underline{\omega}\cdot\underline{\omega})\overrightarrow{AP}$$

$$\omega^2\overrightarrow{AP} = \underline{\omega} \times \underline{V} + (\underline{\omega}\cdot\overrightarrow{AP})\underline{\omega}$$

$$\overrightarrow{AP} = \frac{\underline{\omega} \times \underline{V}}{\omega^2} + \left(\frac{\underline{\omega}\cdot\overrightarrow{AP}}{\omega^2}\right)\underline{\omega}$$

$$\vec{AP} = \frac{\underline{\omega} \times \underline{V}}{\omega^2} + \mu \underline{\omega} \longrightarrow (A)$$

where $\mu = \left(\frac{\underline{\omega} \cdot \vec{AP}}{\omega^2}\right)$ is an

arbitrary scalar because at the instant under consider and for a particular point P it is invariant but any other point the central axis line it will be different.

Thus any point P with velocity parallel to $\underline{\omega}$ lies on the line (A)

Now $\underline{v} = [u, 0, 0]$

$$\underline{\omega} = \frac{\sqrt{3}\,v}{2a}\hat{i} + \frac{v}{2a}\hat{j} + 0\hat{k} \quad \omega^2 = \frac{v^2}{a^2}$$

$$\underline{\omega} \times \underline{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\sqrt{3}\,u}{2a} & \frac{v}{2a} & 0 \\ u & 0 & 0 \end{vmatrix}$$

$$= -\left(\frac{uv}{2a}\right)\hat{k}$$

Using these values in (A)

$$\vec{AP} = \frac{-\left(\frac{uv}{2a}\right)\hat{k}}{\left(\frac{v^2}{a^2}\right)} + \mu\left(\frac{\sqrt{3}\,v}{2a}\hat{i} + \frac{v}{2a}\hat{j} + 0\hat{k}\right)$$

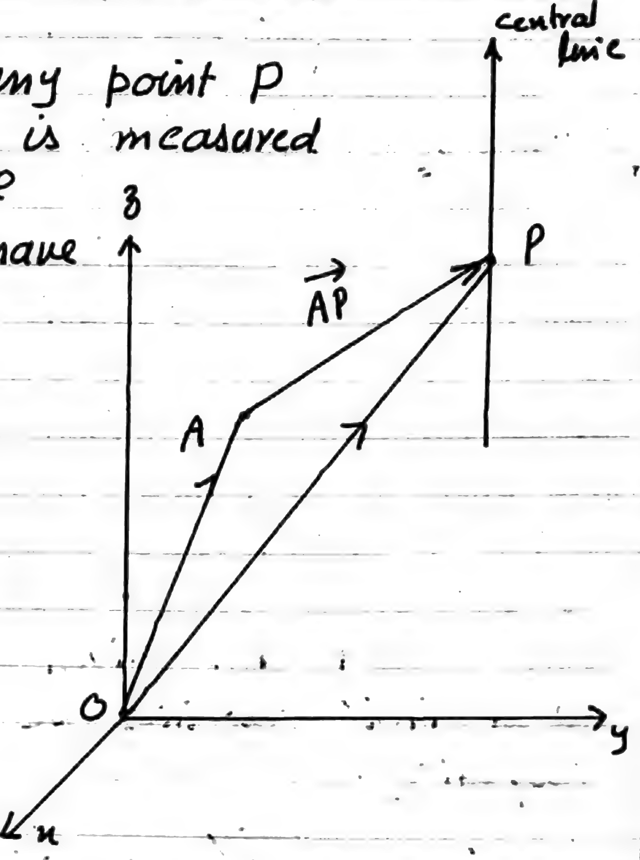
$$= -\frac{ua}{2v}\hat{k} + \mu\left(\frac{\sqrt{3}\,v}{2a}\hat{i} + \frac{v}{2a}\hat{j} + 0\hat{k}\right)$$

$$\vec{AP} = \left(\frac{\mu v\sqrt{3}}{2a}\right)\hat{i} + \left(\frac{\mu v}{2a}\right)\hat{j} - \left(\frac{ua}{2v}\right)\hat{k}$$

This equation of central line when p.v of each point of line is measured relative to A.

Equation of Central line Relative fixed frame

When position of any point P on the central line is measured relative to origin of the fixed axes. we have



$$\overrightarrow{OP} = \overrightarrow{AP} + \overrightarrow{OA}$$

But $\overrightarrow{OA} = [a, 0, 0]$

$$\overrightarrow{AP} = \left(\frac{\mu v\sqrt{3}}{2a}\right)\hat{i} + \left(\frac{\mu v}{2a}\right)\hat{j} - \left(\frac{ua}{2v}\right)\hat{k}$$

$$\Rightarrow \overrightarrow{OP} = [a, 0, 0] + \left[\frac{\mu v\sqrt{3}}{2a}, \frac{\mu v}{2a}, -\frac{ua}{2v}\right]$$

$$= \left(a + \frac{\mu v\sqrt{3}}{2a}\right)\hat{i} + \left(\frac{\mu v}{2a}\right)\hat{j} - \frac{ua}{2v}\hat{k} \rightarrow (B)$$

When this line meets the xz-plane, then point of intersection is common to this line and xz-plane ($y = 0$). So common point will be a such point on line whose y-component is zero. Thus at such point

$$\frac{\mu v}{2a} = 0 \Rightarrow \mu = 0$$

putting $\mu = 0$ in B P.V of point Q intersection is

$$\overrightarrow{OQ} = a\hat{i} + 0\hat{j} - \frac{ua}{2v}\hat{k}$$

$\Rightarrow$ Co-ordinates of point of intersection Q are

$$\left(a\ ,\ \ 0\ ,\ -\frac{ua}{2v}\right)$$

Problem # The points $(a, 2a, -a)$, $(-a, -a, a)$ and (a, a, a) of a rigid body have instantaneous velocities $\left(\frac{\sqrt{3}V}{2}, 0, \frac{\sqrt{3}V}{2}\right)$, $\left(-\frac{V}{\sqrt{3}}, 0, \frac{-V}{\sqrt{3}}\right)$ and $\left(0, -\frac{V}{\sqrt{3}}, \frac{V}{\sqrt{3}}\right)$ respectively w.r.t a rectangular Co-ordinate system. Show that the body has line through the origin having direction Cosines $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

Sol # Let the given points be A, B, C and their velocities be denoted by $\underline{V}_A, \underline{V}_B, \underline{V}_C$ respectively.

$$\underline{V}_A = \left[\frac{\sqrt{3}V}{a}, 0, \frac{\sqrt{3}V}{a}\right]$$

$$\underline{V}_B = \left[-\frac{V}{\sqrt{3}}, 0, \frac{-V}{\sqrt{3}}\right]$$

$$\underline{V}_C = \left[0, -\frac{V}{\sqrt{3}}, \frac{V}{\sqrt{3}}\right]$$

Taking A as the reference point.

$$\underline{r}_1 = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = [-a, -a, a] - [a, 2a, -a]$$

$$= [-2a, -3a, 2a]$$

$$\underline{r}_2 = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \left[a, \cancel{+\frac{V}{\sqrt{3}}}^{\,a}, \cancel{\frac{V}{\sqrt{3}}}^{\,a}\right] - [a, 2a, -a]$$

$$= [0, -a, 2a]$$

Velocity of B relative to A

$$\underline{V}_1 = \underline{V}_{B/A} = \underline{V}_B - \underline{V}_A = \left[\frac{-V}{\sqrt{3}}, 0, \frac{-V}{\sqrt{3}}\right] - \left[\frac{\sqrt{3}V}{2}, 0, \frac{\sqrt{3}V}{2}\right]$$

$$= \left[-\frac{5V}{2\sqrt{3}}, 0, \frac{-5V}{2\sqrt{3}}\right]$$

$$\underline{V_2} = \underline{V_{C/A}} = \underline{V_C} - \underline{V_A}$$

$$= \left[0, \frac{-V}{\sqrt{3}}, \frac{V}{\sqrt{3}}\right] - \left[\frac{\sqrt{3}}{2}V, 0, \frac{\sqrt{3}V}{2}\right]$$

$$= \left[-\frac{\sqrt{3}V}{2}, -\frac{V}{\sqrt{3}}, -\frac{V}{2\sqrt{3}}\right]$$

Let $\underline{\omega} = [\omega_1, \omega_2, \omega_3]$ be the angular velocity of rigid body.

$$\underline{V_1} = \underline{\omega} \times \underline{r_1}$$

$$\left[-\frac{5V}{2\sqrt{3}}, 0, \frac{-5V}{2\sqrt{3}}\right] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ -2a & -3a & 2a \end{vmatrix}$$

$$= \hat{i}(2a\omega_2 + 3a\omega_3) - \hat{j}(2a\omega_1 + 2a\omega_3) + \hat{k}(-3a\omega_1 + 2a\omega_2)$$

$$\Rightarrow 2a\omega_2 + 3a\omega_3 = -\frac{5V}{2\sqrt{3}} \longrightarrow (1)$$

$$-(2a\omega_1 + 2a\omega_3) = 0$$

$$\omega_1 = -\omega_3 \longrightarrow (2)$$

$$-3a\omega_1 + 2a\omega_2 = -\frac{5V}{2\sqrt{3}} \longrightarrow (3)$$

$$\underline{V_2} = \underline{\omega} \times \underline{r_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ 0 & -a & 2a \end{vmatrix}$$

$$\left[-\frac{\sqrt{3}V}{2}, -\frac{V}{\sqrt{3}}, -\frac{V}{2\sqrt{3}}\right] = \hat{i}(2a\omega_2 + a\omega_3) - \hat{j}(2a\omega_1) + \hat{k}(-a\omega_1)$$

$$\Rightarrow \quad 2a\omega_2 + a\omega_3 = -\frac{\sqrt{3}}{2}V \longrightarrow (4)$$

$$-2a\omega_1 = -\frac{V}{\sqrt{3}}$$

$$\omega_1 = \frac{V}{2a\sqrt{3}}$$

from (2)

$$\omega_3 = -\frac{v}{2a\sqrt{3}}$$

from (4) by putting value of ω_3

$$2a\omega_2 - \frac{v}{2\sqrt{3}} = -\frac{\sqrt{3}}{2}V$$

$$2a\omega_2 = \frac{v}{2\sqrt{3}} - \frac{\sqrt{3}}{2}V$$

$$= \frac{v - 3V}{2\sqrt{3}} = \frac{-2V}{2\sqrt{3}}$$

$$\omega_2 = -\frac{V}{2a\sqrt{3}}$$

So $\underline{\omega} = [\omega_1, \omega_2, \omega_3]$

$$= \left[\frac{V}{2a\sqrt{3}}, \frac{-V}{2a\sqrt{3}}, \frac{-V}{2a\sqrt{3}}\right]$$

If P is any point on the central line, then equation of central line w.r.t base point A is given by

$$\overrightarrow{AP} = \frac{\underline{\omega} \times \underline{V}_A}{\omega^2} + \mu\,\underline{\omega} \longrightarrow (5)$$

Now $\underline{V}_A = \left[\frac{\sqrt{3}}{2}V, 0, \frac{\sqrt{3}}{2}v\right]$

$\underline{\omega} = \left[\frac{v}{2a\sqrt{3}}, \frac{-v}{2a\sqrt{3}}, \frac{-v}{2a\sqrt{3}}\right]$ as calculated above

$$\omega^2 = \frac{v^2}{12a^2} + \frac{v^2}{12a^2} + \frac{v^2}{12a^2} = \frac{v^2}{4a^2}$$

$$\underline{\omega} \times \underline{v}_A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{v}{2a\sqrt{3}} & \frac{-v}{2a\sqrt{3}} & \frac{-v}{2a\sqrt{3}} \\ \frac{\sqrt{3}}{2}v & 0 & \frac{\sqrt{3}}{2}v \end{vmatrix}$$

$$= -\frac{v^2}{4a}\hat{i} - \frac{2v^2}{4a}\hat{j} + \frac{v^2}{4a}\hat{k}$$

using these values in ⑤.

$$\overrightarrow{AP} = -a\hat{i} - 2a\hat{j} + a\hat{k} + \mu\frac{v}{2a\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$$

Now equation of central line w.r.t fixed frame is

$$\overrightarrow{OP} = \overrightarrow{AP} + \overrightarrow{OA}$$

$$= (a\hat{i} + 2a\hat{j} - a\hat{k}) + (-a\hat{i} - 2a\hat{j} + a\hat{k}) + \mu\frac{v}{2a\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$$

$$= \frac{\mu v}{2a\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$$

Comparing it with equation of a line passing through origin viz

$$\underline{r} = t\,\underline{b}$$

We note that this central line of body

passes through origin and is parallel to vector

$$\hat{i} - \hat{j} - \hat{k}$$

which is called direction vector of the line

$\because$ Direction Cosines of line are same as that of its D-vector

$\therefore$ D-Cosines of the line are

$$\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$$

Thus a the body has a line passing through origin with D-Cosines $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$
Hence proved.

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Problem # A rigid body is rotating about fixed origin O, the points A(0,-1,2) and B(2,0,0) are moving with velocities [7,-2,-1] [0, 6, -4] respectively. Find the angular velocity of the body if units used are meters and seconds.

Sol # Let p-vectors of points A & B be denoted by $\underline{r}_A$ & $\underline{r}_B$ respectively. Then

$$\underline{r}_A = \overrightarrow{OA} = [0, -1, 2]$$

$$\underline{r}_B = \overrightarrow{OB} = [2, 0, 0]$$

Let the given velocities of A & B be

$$\underline{V}_A = [7, -2, -1]$$

$$\underline{V}_B = [0, 6, -4]$$

$$\underline{V}_A = \underline{\omega} \times \underline{r}_A$$

$$\Rightarrow [7, -2, -1] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= \hat{i}(2\omega_2 + \omega_3) - \hat{j}(2\omega_1) + \hat{k}(-\omega_1)$$

$\Rightarrow \omega_1 = 1$

$2\omega_2 + \omega_3 = 7 \longrightarrow ①$

Also

$$\underline{V}_B = \underline{\omega} \times \underline{r}_B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ 2 & 0 & 0 \end{vmatrix}$$

$$\Rightarrow [0, 6, -4] = 0\hat{i} - \hat{j}(-2\omega_3) + \hat{k}(-2\omega_2)$$

$\Rightarrow \quad 2\omega_3 = 6 \Rightarrow \omega_3 = 3$

and $-2\omega_2 = -4 \Rightarrow \omega_2 = 2.$

so angular velocity of the body is

$$\underline{\omega} = [1, 2, 3]$$

Problem# A particle describes a circle about a line whose equation in vector form is given by

$$\underline{r} = (3+\lambda)\hat{i} + 2\hat{j} + (1-\lambda)\hat{k}$$

with angular speed ω radians/sec. If at $t=0$ the particle is at point $2\hat{k}$, find its velocity

Sol# The equation of the axis of rotation is

$$\underline{r} = (3+\lambda)\hat{i} + 2\hat{j} + (1-\lambda)\hat{k}$$

$$\underline{r} = (3\hat{i}+2\hat{j}+\hat{k}) + \lambda(\hat{i}+\hat{k})$$

$$= \underline{a} + \lambda\,\underline{b}$$

where $\underline{a} = 3\hat{i}+2\hat{j}+\hat{k}$ $\quad \underline{b} = \hat{i}+\hat{k}$

$\Rightarrow$ axis is a line passing through point $\underline{a}$ and parallel to a vector $\underline{b}$

Let A be fixed point with p.v $\underline{a}$.

$\because$ $\underline{\omega}$ is parallel to axis of rotation

$\therefore$ $\underline{\omega}$ is parallel to $\underline{b}$

D. cosines and unit vector along $\underline{\omega}$ are same as those of $\underline{b}$

unit vector along $\underline{\omega}$ or axis of rotation

$$= \hat{b} = \frac{\underline{b}}{|\underline{b}|} = \frac{\hat{i}+\hat{k}}{\sqrt{2}}$$

$$\underline{\omega} = \omega\hat{b} = \frac{\omega}{\sqrt{2}}(\hat{i}+\hat{k})$$

Let position of particle at $t=0$ be at point P with p.v $2\hat{k}$ i.e

$$\overrightarrow{OP} = 2\hat{k}$$

Now $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$$= 2\hat{k} - \underline{a}$$

$$= 2\hat{k} - (3\hat{i}+2\hat{j}+\hat{k})$$

$$= -3\hat{i}-2\hat{j}+\hat{k}$$

So velocity of particle at point P is

$$\underline{V} = \underline{\omega} \times \overrightarrow{AP}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\omega}{\sqrt{2}} & 0 & -\frac{\omega}{\sqrt{2}} \\ -3 & -2 & 1 \end{vmatrix}$$

$$\underline{V} = \hat{i}\left(-\frac{2}{\sqrt{2}}\omega\right) - \hat{j}\left(\frac{\omega}{\sqrt{2}} + \frac{3}{\sqrt{2}}\omega\right) + \hat{k}\left(-\frac{2}{\sqrt{2}}\omega\right)$$

$$= \hat{i}(-\sqrt{2}\omega) - \hat{j}\left(-\frac{2\omega}{\sqrt{2}}\right) - \sqrt{2}\hat{k}\cdot\omega$$

$$= -\sqrt{2}\omega\hat{i} + \sqrt{2}\omega\hat{j} - \sqrt{2}\omega\hat{k}$$

$$= -\sqrt{2}\omega(\hat{i} - \hat{j} + \hat{k}$$

Problem # A rigid body receives three successive rotations about three mutually perpendicular intersecting lines fixed in space. Each rotation is through right angle and in the senses being cyclic. Find the axis and magnitude of single equivalent rotation.

Sol # We know that displacement of body due to a rotation $\delta\theta$ about an axis with unit vector $\hat{n}$ is given by

$$d\underline{r} = d\theta\,\hat{n} \times \underline{r}$$

Since rotation of rigid can be specified by the position of single particle of it, therefore let $\underline{r}$ be initial position of a particle of the body. Let the three mutually $\perp$ar axis of rotation fixed in space be x-axis y-axis, z-axis fixed at point O. Let $\underline{r}_1$ $\underline{r}_2$, $\underline{r}_3$ be p.Vs of particle recived by it after rotations about these axes.

Then

$$\underline{r}_1 = \underline{r} + d\theta\,\hat{n} \times \underline{r}$$

$$= \underline{r} + \frac{\pi}{2}\hat{i} \times \underline{r} \longrightarrow (1)$$

After this rotation p.v of particle is $\underline{r}_1$

Now after 2nd rotation about y-axis P.V of particle $\underline{r}_2$ is

$$\underline{r}_2 = \underline{r}_1 + d\theta\,\hat{n} \times \underline{r}_1$$

$$= \underline{r}_1 + \frac{\pi}{2}\hat{j} \times \underline{r}_1$$

$$= \left(\underline{r} + \frac{\pi}{2}\hat{i}\times\underline{r}\right) + \frac{\pi}{2}\hat{j}\times\left(\underline{r} + \frac{\pi}{2}\hat{i}\times\underline{r}\right)$$

$$= \underline{r} + \frac{\pi}{2}\hat{i}\times\underline{r} + \frac{\pi}{2}\hat{j}\times\underline{r} + \left(\frac{\pi}{2}\right)^2\hat{j}\times(\hat{i}\times\underline{r})$$

Direct Method

Let $\underline{r}$ be initial position of a particle and $d\underline{\theta}_1$, $d\underline{\theta}_2$, $d\underline{\theta}_3$ be rotations about x-axis, y-axis and z-axis.

Then
$$d\underline{\theta}_1 = \frac{\pi}{2}\hat{i}$$
$$d\underline{\theta}_2 = \frac{\pi}{2}\hat{j}$$
$$d\underline{\theta}_3 = \frac{\pi}{2}\hat{k}$$

If $d\underline{r}$ is net change in $\underline{r}$ due all of these rotations, then

$$d\underline{r} = (d\underline{\theta}_1 + d\underline{\theta}_2 + d\underline{\theta}_3)\times\underline{r}$$

$$= \frac{\pi}{2}(\hat{i}+\hat{j}+\hat{k})\times\underline{r} \longrightarrow (2)$$

If $\underline{\omega}$ is net angular velocity due to all three rotations, then

$$d\underline{r} = \underline{\omega}\,dt \times \underline{r}$$

$$\Rightarrow \quad \underline{\omega}\,dt = \frac{\pi}{2}(\hat{i}+\hat{j}+\hat{k})$$

$$\underline{\omega} = \frac{\pi}{2}\left(\frac{d\hat{i}}{dt} + \frac{d\hat{j}}{dt} + \frac{d\hat{k}}{dt}\right)$$

From ② net rotation is given by

$$d\underline{\theta} = \frac{\pi}{2}(\hat{i}+\hat{j}+\hat{k})$$

Its magnitude is given by

$$d\theta = \frac{\pi}{2}\sqrt{3} = \frac{\sqrt{3}}{2}\pi$$

Problem # A body is pivoted at a point o is rotating at the rate of 90 radian per second about a fixed line in the direction of a vector [-1, 2, 2] to an observer looking in the direction of this vector. The sense of rotation of the body is clockwise. Find vel of P with p.v $(1/3, 2/3, -1/3)$

Sol # unit vector in the direction of axis and angular velocity is

$$\hat{a} = \frac{-\hat{i}+2\hat{j}+2\hat{k}}{3}$$

so direction of the angular velocity is along $\hat{a}$

$$\underline{\omega} = \omega\hat{a}$$

$$= 90\cdot\left(\frac{-\hat{i}+2\hat{j}+2\hat{k}}{3}\right)$$

$$= 30(-\hat{i}+2\hat{j}+2\hat{k})$$

P.V of point P relative to fixed pivot O is

$$\underline{r} = \overrightarrow{OP} = [1/3, 2/3, -1/3]$$

$$\underline{V}_p = \underline{\omega}\times\underline{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -30 & 60 & 60 \\ 1/3 & 2/3 & -1/3 \end{vmatrix}$$

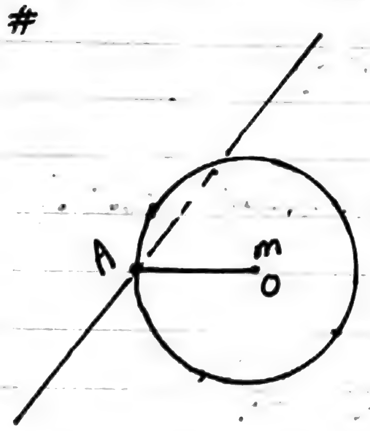
$$= \hat{i}(-20-40) - \hat{j}(-20+10) + \hat{k}(-20-20)$$

$$= -60\hat{i} + 10\hat{j} - 40\hat{k}$$

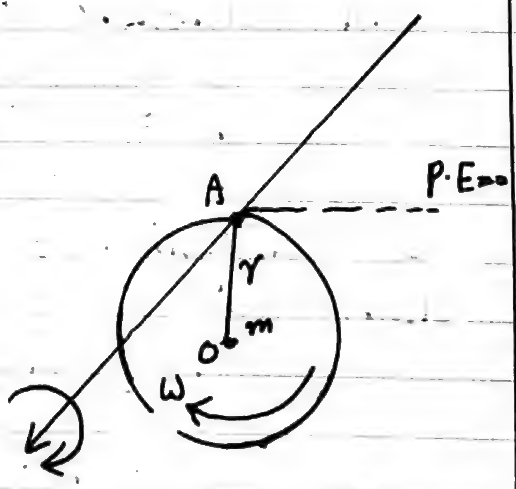
$$= 10(-6\hat{i} + \hat{j} - 4\hat{k})$$

Problem # A uniform circular disc of mass m, radius r and centre O is free to turn in its own plane about a smooth horizontal axis passing through a pt A on the rim of the disc. The disc is released from rest in the position in which OA is horizontal and the disc is vertical. Find the angular velocity of the disc when OA 1st becomes vertical.

Sol #



Initial position

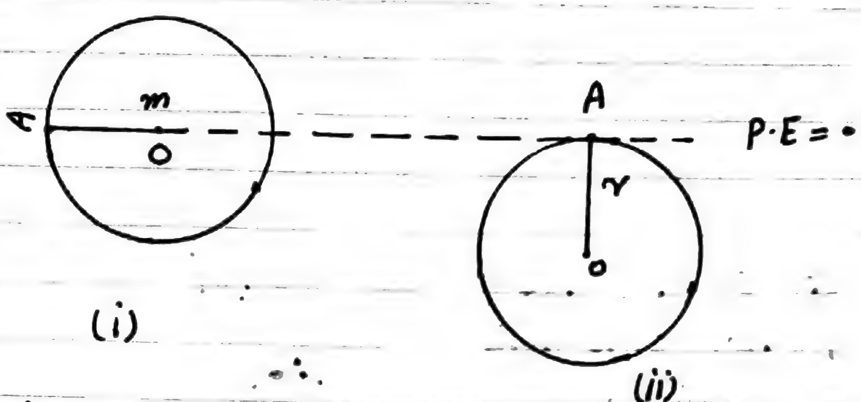


Position with AO vertical.

The disc will rotate in a vertical plane since the axis of rotation is horizontal. The moment of inertia of the disc about the axis through A perpendicular to the disc is given by

$$I = \frac{1}{2}mr^2 + mr^2 \quad \text{||al axis Theorem}$$

$$I = 3mr^2/2$$



Initially diagram (i)

$K.E = 0$ $\qquad$ $P.E = 0$ because total mass is on horizontal level

total initial energy $= 0$.

When OA is vertical diagram (ii)

$$K.E = \frac{1}{2} I \omega^2 = \frac{1}{2}\left(\frac{3mr^2}{2}\right)\omega^2$$

$$P.E = -mgr$$

By principal of conservation of Mechanical energy, we have

$$0 = -mgr + \frac{3}{4} mr^2 \omega^2$$

$\omega = 2\sqrt{g/3r}$ is required angular velocity of the disc.

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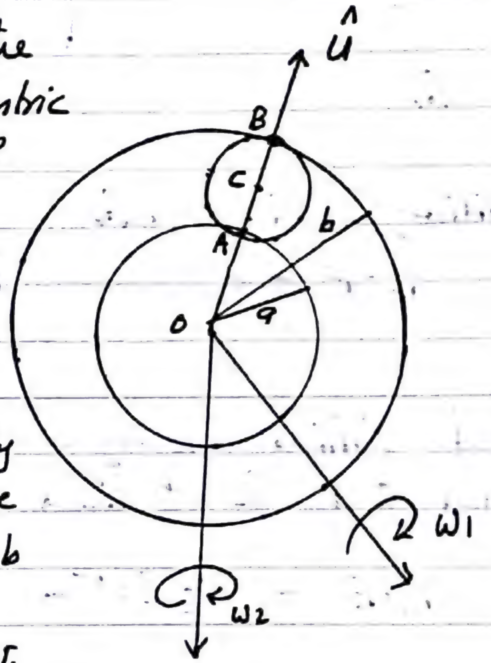
Govt. College Asghar Mall RWP #

Problem # Two concentric spherical surfaces of radii a, b rotate with angular velocities $\underline{\omega_1}$, ω_2 respectively, about diameters inclined to one another at angle α. A sphere placed between them rolls in contact with both. Prove that the centre of sphere describes a circle with angular velocity

$$\frac{(a^2\omega_1^2+b^2\omega_2^2+2ab\,\omega_1\omega_2\cos\alpha)^{1/2}}{a+b}$$

Sol # Let O be the centre of two concentric spheres of radii a & b and sphere placed between them have angular velocity $\underline{\Omega}$.



Suppose instantaneously the sphere touches the spheres of radii a & b at points A and B respectively and C is its centre.

$$OA = a \qquad OB = b$$

Considering A as point of the sphere of radius a, we have

$$\underline{V}_A = \underline{V}_O + \underline{\omega}_1 \times \overrightarrow{OA}$$

But $\underline{V}_O = \underline{0}$ because the centre of surfaces is fixed.

$$\underline{V}_A = \underline{\omega}_1 \times \underline{OA}$$

Considering A as a point of sphere with centre C, we have velocity of A

$$\underline{V}_C + \underline{\Omega} \times \underline{CA}$$

But since the rolling takes place at A, there

velocities must be equal. so

$$\underline{\omega_1} \times \underline{OA} = \underline{V_c} + \underline{\Omega} \times \underline{CA} \longrightarrow ①$$

Similarly for point B as point of circle of radius b, we have

$$\underline{V_B} = \underline{\omega_2} \times \underline{OB}$$

and as a point of circle with centre at c velocity of B is

$$\underline{V_c} + \underline{\Omega} \times \underline{CA}$$

$\Rightarrow$

$$\underline{\omega_2} \times \underline{OB} = \underline{V_c} + \underline{\Omega} \times \underline{CA} \longrightarrow ②$$

Adding ① and ②

$$2\underline{V_c} = \underline{\omega_1} \times \underline{OA} + \underline{\omega_2} \times \underline{OB}$$

Now

$$\underline{OA} = a\hat{u} \qquad \underline{OB} = b\hat{u}$$

$$2\underline{V_c} = a\omega_1 \times \hat{u} + b\omega_2 \times \hat{u}$$

$$= (a\omega_1 + b\underline{\omega_2}) \times \hat{u}$$

$$\Rightarrow \underline{V_c} = \left(\frac{a\omega_1 + b\omega_2}{2}\right) \times \hat{u}$$

$$= \left(\frac{a\omega_1 + b\omega_2}{a+b}\right) \times \left(\frac{a+b}{2}\hat{u}\right)$$

Now $\frac{a+b}{2} = OC$

because

$$OB = a + AB$$

$$OC + CB = a + b - a$$

$$OC + \frac{AB}{2} = a + AB$$

$$OC = a + \frac{AB}{2} = a + \frac{b-a}{2}$$

$$= \frac{a+b}{2}$$

$$\overrightarrow{OC} = \frac{a+b}{2}\hat{u}$$

Thus $\underline{v}_c = \left(\frac{a\underline{\omega}_1 + b\underline{\omega}_2}{a+b}\right) \times \vec{OC}$

$\Rightarrow \quad \underline{v}_c = \underline{\omega} \times \vec{OC}$

where $\underline{\omega} = \frac{a\underline{\omega}_1 + b\underline{\omega}_2}{a+b}$

Now let $\underline{r}_c$ be p.v of point C relative to O. Then

$$\underline{v}_c = \frac{d\underline{r}_c}{dt}$$

$$\Rightarrow \quad \frac{d\underline{r}_c}{dt} = \underline{\omega} \times \vec{OC}$$
$$= \underline{\omega} \times \underline{r}_c \quad \rightarrow (a)$$

Taking dot product of this with $\underline{r}_c$

$$\underline{r}_c \cdot \frac{d\underline{r}_c}{dt} = \underline{r}_c \cdot \underline{\omega} \times \underline{r}_c = 0$$

$$\underline{r}_c \cdot \frac{d\underline{r}_c}{dt} = 0 \quad \rightarrow (3)$$

But $\frac{d}{dt}(\underline{r}_c \cdot \underline{r}_c) = \underline{r}_c \cdot \frac{d\underline{r}_c}{dt} + \frac{d\underline{r}_c}{dt} \cdot \underline{r}_c$

$$= 2\underline{r}_c \cdot \frac{d\underline{r}_c}{dt}$$

$$\frac{d}{dt}(r_c^2) = 2\underline{r}_c \cdot \frac{d\underline{r}_c}{dt}$$

$$= 0$$

Integrating

$$r_c^2 = \text{Constat} =$$

let $r_c^2 = d^2 \Rightarrow r_c = d$

$\Rightarrow$ ~~C describes a~~ locus of C lies on a sphere with centre at O and radius d

from (a)

$$\hat{\omega} \cdot \frac{d\underline{r}_c}{dt} = \hat{\omega} \cdot \underline{\omega} \times \underline{r}_c = 0$$

$$\Rightarrow \frac{d}{dt}(\hat{\omega}\cdot \underline{r}_c) = 0$$

$$\Rightarrow \hat{\omega}\cdot \underline{r}_c = \text{Constant} = h \text{ (say)} \longrightarrow (b)$$

which is equation of plane. This shows that locus of c also lies on a plane

Thus locus of c is intersection of this plane and sphere of radius d

Thus C describes a circle with constant angular velocity

$$\underline{\omega} = \frac{a\underline{\omega}_1 + b\underline{\omega}_2}{a+b}$$

$$\omega^2 = \underline{\omega}\cdot\underline{\omega} = \left(\frac{a\underline{\omega}_1 + b\underline{\omega}_2}{a+b}\right)\cdot\left(\frac{a\underline{\omega}_1 + b\underline{\omega}_2}{a+b}\right)$$

$$= \frac{1}{(a+b)^2}\left[a^2\omega_1^2 + ab\,\underline{\omega}_1\cdot\underline{\omega}_2 + ab\,\underline{\omega}_1\cdot\underline{\omega}_2 + b^2\omega_2^2\right]$$

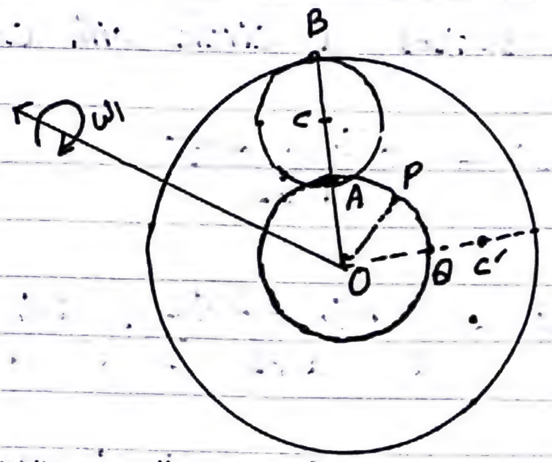
$$= \frac{1}{(a+b)^2}\left[a^2\omega_1^2 + b^2\omega_2^2 + 2ab\,\omega_1\omega_2\cos\alpha\right]$$

$$\omega = \frac{\left(a^2\omega_1^2 + b^2\omega_2^2 + 2ab\,\omega_1\omega_2\cos\alpha\right)^{1/2}}{a+b}$$

Problem*# (2000) A circular cylinder of radius "a" is enclosed inside a co-axial hollow circular cylinder of radius (b) $(b > a)$. The space between them contains the ball bearing of radius $\frac{1}{2}(b-a)$. The inner and the outer cylinders are made to turn with constant angular speeds ω_1 and ω_2 respectively. Show that if there is no slipping the angular speed of a ball is $\frac{\omega_2 b - \omega_1 a}{b-a}$ and that the centre of a ball moves in a circle with angular

angular speed $\frac{\omega_1 a+\omega_2 b}{a+b}$. What will be the length of arc of contact with either cylinder in unit time:

Sol #



Suppose that the rotations are clockwise. Let ω be the angular speed of the ball bearing, ω_c the angular speed of radius OC. Let A & B be the points of contact of ball bearing with inner and outer cylinder at any time t

For Rolling at Point B

Considering point B as a point of outer cylinder of radius b as a point of ball bearing

$$V_B = V_A + \omega AB \rightarrow ①$$ B is considered a point of ball bearing

Considering B as a point of outer cylinder its velocity is

$$\omega_2 b$$

using in ①

$$\omega_2 b = V_A + \omega(b-a) \rightarrow ②$$ $\because AB = b-a$

By considering A is a point of inner cylinder of radius a its velocity is

$$V_A = \omega_1 a$$

using in ②

$$\omega_2 b = \omega_1 a + \omega(b-a)$$

$$\omega = \frac{\omega_2 b - \omega_1 a}{b-a}$$

The period of revolution of ball bearing is

$$T = \frac{2\pi}{|\omega|}$$

$$= \frac{2\pi(b-a)}{|\omega_2 b - \omega_1 a|}$$

Velocity of the centre C of ball bearing is

$$V_C = V_A + \omega AC$$

$$= \omega_1 a + \frac{\omega_2 b - \omega_1 a}{b-a} \cdot AC$$

$$AC = \frac{AB}{2} = \frac{OB - OA}{2} = \frac{b-a}{2}$$

$$V_C = \omega_1 a + \frac{\omega_2 b - \omega_1 a}{b-a} \cdot \frac{b-a}{2}$$

$$= \frac{\omega_1 a + \omega_2 b}{2}$$

Considering C as a point of ball bearing

$$V_C = \omega_c OC$$

$$\omega_c = \frac{V_C}{OC}$$

But $OC = OB - BC = b - \frac{1}{2}(b-a)$

$$= \frac{a+b}{2}$$

$$\omega_c = \frac{\omega_1 a + \omega_2 b}{2} \times \frac{2}{a+b} = \frac{\omega_1 a + \omega_2 b}{a+b}$$

Hence moves in a circle of radius oc with time period

$$T' = \frac{2\pi}{|\omega_c|} = \frac{2\pi(a+b)}{\omega_1 a + \omega_2 b}$$

Length of the arc of Contact

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In unit time radius of inner cyliner Covers angular displacement ω_1 ang goes from position OA to OP while in unit time oc Covers angle ω_c and goes from OC to OC'

Relative angular distance $|\omega_c - \omega_1|$

Length of the arc of contact is arc length covered during relative angular distance $|\omega_c - \omega_1|$ by radius (a) of inner cylineder and this is equal to $\widehat{PQ}$. Thus

$$\widehat{PQ} = a|\omega_c - \omega_1|$$

$$= a\left|\frac{\omega_1 a + \omega_2 b}{a+b} - \omega_1\right|$$

$$= \frac{ab|\omega_2 - \omega_1|}{a+b}$$

Problem # A plane which is fixed in space has equation

$$lx + my + nz + p = 0$$

referred to a set of orthogonal axes OXYZ which rotate with angular velocity $\underline{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$, where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ respectively. Prove

that

$$\dot{l} = \omega_3 m - \omega_2 n$$
$$\dot{m} = \omega_1 n - \omega_3 l$$
$$\dot{n} = \omega_2 l - \omega_1 m$$

where the dot denotes differentiation w.r.t time.

Sol # Note general equation of plane can be written as

$$ax + by + cz + d = 0$$

where $[a, b, c]$ is vector perpendicular to the plane

Here given plane is

$$lx + my + nz + p = 0$$

A vector $\underline{u}$ perpendicular to the plane referred to the axes $O(XYZ)$ is

$$\underline{u} = l\hat{i} + m\hat{j} + n\hat{k}$$

Let $\left(\frac{d}{dt}\right)_s$ denotes derivative w.r.t space axes

and $\left(\frac{d}{dt}\right)_r$ denotes derivative w.r.t rotating axes

$O(XYZ)$. Then

$$\left(\frac{d\underline{u}}{dt}\right)_s = \left(\frac{d\underline{u}}{dt}\right)_r + \underline{\omega} \times \underline{u}$$

But $\underline{u}$ is constant in space axes because plane is fixed in space.

Therefor $\left(\frac{d\underline{u}}{dt}\right)_s = \underline{0}$

$$\Rightarrow \quad \underline{0} = \dot{l}\hat{i} + \dot{m}\hat{j} + \dot{n}\hat{k} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ l & m & n \end{vmatrix}$$

$$0 = \dot{l}\hat{i} + \dot{m}\hat{j} + \dot{n}\hat{k} + \hat{i}(n\omega_2 - m\omega_3) - \hat{j}(n\omega_1 - l\omega_3) + \hat{k}(m\omega_1 - l\omega_2)$$

$$0 = (\dot{l} + n\omega_2 - m\omega_3)\hat{i} + (\dot{m} - n\omega_1 + l\omega_3)\hat{j} + (\dot{n} + m\omega_1 - l\omega_2)\hat{k}$$

Comparing co-efficients of $\hat{i}, \hat{j}, \hat{k}$, we have

$$\dot{l} = m\omega_3 - n\omega_2 \rightarrow ①$$

$$\dot{m} = n\omega_1 - l\omega_3 \rightarrow ②$$

$$\dot{n} = l\omega_2 - m\omega_1 \rightarrow ③$$

<u>Note</u> Multiplying ① ② & ③ with l, m, n and adding, we have

$$l\dot{l} + m\dot{m} + n\dot{n} = 0$$

This result can be obtained directly as

$$u^2 = l^2 + m^2 + n^2$$

Diff w.r.t t

$$\frac{du^2}{dt} = 2l\dot{l} + 2m\dot{m} + 2n\dot{n}$$

But u is constant

$$\therefore \frac{du^2}{dt} = 0$$

$$\Rightarrow 2l\dot{l} + 2m\dot{m} + 2n\dot{n} = 0$$

$$l\dot{l} + m\dot{m} + n\dot{n} = 0$$

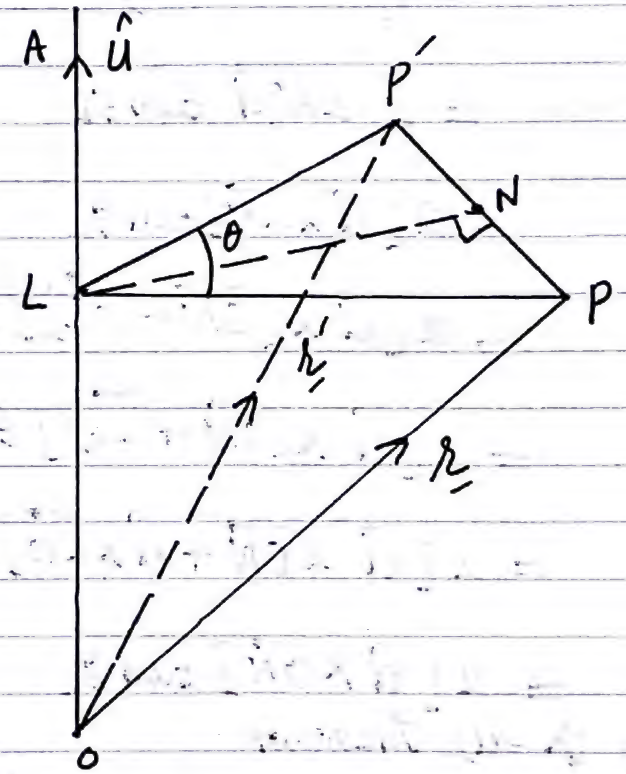
Problem# An origin O is taken on a fixed axis OA and P is any point not on OA. The plane POA is rotated about OZ through an angle θ into position P′OZ. Prove that

$$\underline{r}' - \underline{r} = \hat{u} \times (\underline{r}' + \underline{r}) \tan \theta/2$$

where $\overrightarrow{OP} = \underline{r}$, $\overrightarrow{OP'} = \underline{r}'$ and $\hat{u}$ is unit vector along $\overrightarrow{OA}$.

Hence prove that the velocity of P is $\underline{\omega} \times \underline{r}$, where $\underline{\omega} = \dot{\theta}\hat{u}$

Sol#



Draw PL to OA. Let N be mid point of PP′. The angle PLP′ is equal to θ.

$$\because \overline{LP} = \overline{LP'}$$

$\therefore$ $\Delta PP'L$ is an isosceles triangle

$\Rightarrow$ LN bisects the angle PLP′

$$\underline{r}' - \underline{r} = \overrightarrow{OP'} - \overrightarrow{OP} = \overrightarrow{PP'}$$

$PP' = 2PN = 2LN \tan\theta/2$

$\because$ $\hat{u}$ is $\perp$ar to LN

$\therefore$ $\hat{u} \times \overrightarrow{LN}$ is $\perp$ar to OLP

and is parallel to $\overrightarrow{PP'}$.

Let $\hat{a}$ be unit vector along these

Then

$$\overrightarrow{PP'} = |\overrightarrow{PP'}|\,\hat{a}$$

$$= (2LN \tan\theta/2)\,\hat{a}$$

$$= 2\,LN\,\hat{a}\,\tan\theta/2$$

$\hat{u} \times \overrightarrow{LN} = |\hat{u}||\overrightarrow{LN}|\,\hat{a} = LN\hat{a}$

$$= 2[\hat{u} \times \overrightarrow{LN}]\tan\theta/2$$

$$= 2[\hat{u} \times (\overrightarrow{ON} - \overrightarrow{OL})]\tan(\theta/2)$$

$$= 2[\hat{u} \times (\overrightarrow{ON} + \overrightarrow{LO})]\tan\theta/2$$

$$= 2[\hat{u} \times \overrightarrow{ON} + \hat{u} \times \overrightarrow{LO}]\tan\theta/2$$

$$= 2[\hat{u} \times \overrightarrow{ON}]\tan\theta/2 \qquad \because \hat{u} \times \overrightarrow{LO} = \underline{0}$$

By $(\lambda - \mu)$ Theorem

$$(1+1)\overrightarrow{ON} = 1\cdot\overrightarrow{OP} + 1\cdot\overrightarrow{OP'}$$

$$2\overrightarrow{ON} = \overrightarrow{OP} + \overrightarrow{OP'} = \underline{r} + \underline{r}'$$

Hence $\overrightarrow{PP'} = 2[\hat{u} \times \overrightarrow{ON}]\tan\theta/2$

$$\underline{r}' - \underline{r} = 2[\hat{u} \times (\tfrac{\underline{r} + \underline{r}'}{2})]\tan\theta/2$$

Suppose the time interval taken for p to move to p' is Δt. Then

$$\frac{\underline{r}' - \underline{r}}{\Delta t} = \frac{2[\hat{u} \times (\frac{\underline{r} \times \underline{r}'}{2})]\tan(\theta/2)}{\Delta t}$$

$$\frac{\underline{r}'-\underline{r}}{\Delta t} = \hat{u} \times \frac{\underline{r}+\underline{r}'}{2} \cdot \frac{\tan\theta/2}{\theta/2} \cdot \frac{\theta}{\Delta t}$$

In limiting case when

$$\Delta t \to 0 \qquad \theta \to 0 \qquad \frac{\underline{r}+\underline{r}'}{2} \to \underline{r}$$

$$\lim_{\Delta t \to 0} \frac{\underline{r}'-\underline{r}}{\Delta t} = \lim_{\Delta t \to 0} \left[\hat{u} \times \frac{\underline{r}+\underline{r}'}{2} \right] \cdot \frac{\tan\frac{\theta}{2}}{\theta/2} \cdot \frac{\theta}{\Delta t}$$

$$\frac{d\underline{r}}{dt} = \hat{u} \times \underline{r}\,\dot{\theta}$$

$$= \hat{u}\dot{\theta} \times \underline{r}$$

$$\frac{d\underline{r}}{dt} = \underline{\omega} \times \underline{r} \qquad \text{Proved.}$$

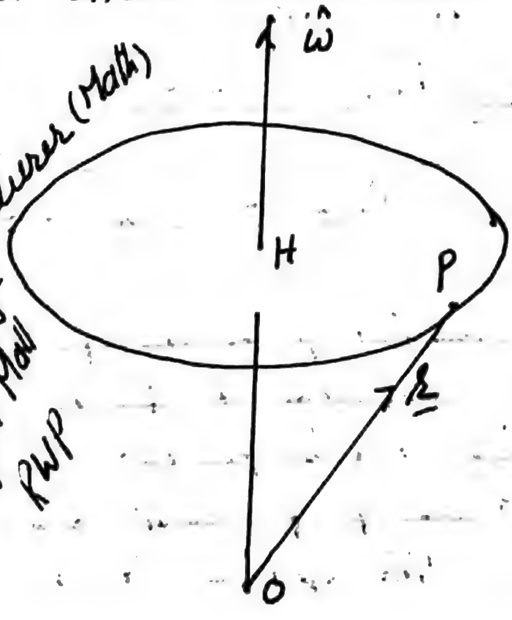
where $\underline{\omega} = \dot{\theta}\hat{u}$

Problem # A point P moves so that its position vector $\underline{r}$, relative to another point O satisfies the equation

$$\frac{d\underline{r}}{dt} = \underline{\omega} \times \underline{r}$$

where $\underline{\omega}$ is a constant vector. Prove that P describes a circle with constant velocity

Sol



We have

$$\frac{d\underline{r}}{dt} = \underline{\omega} \times \underline{r} \longrightarrow ①$$

Dot multiplying with $\underline{r}$

$$\underline{r} \cdot \frac{d\underline{r}}{dt} = \underline{r} \cdot \underline{\omega} \times \underline{r} = 0$$

$$\Rightarrow \quad 2\underline{r} \cdot \frac{d\underline{r}}{dt} = 0$$

$$\Rightarrow \quad \frac{d}{dt}(\underline{r} \cdot \underline{r}) = 0$$

$$\frac{d}{dt}(r^2) = 0$$

Integrating $\quad r^2 = \text{a constant}$

let

$$r^2 = a^2$$

Hence the locus of p lies on sphere of radius (a) and centre at (O)

Dot multiplying ① with $\hat{\omega}$

$$\hat{\omega} \cdot \frac{d\underline{r}}{dt} = \hat{\omega} \cdot \underline{\omega} \times \underline{r} = 0$$

$$\Rightarrow \hat{\omega} \cdot \frac{d\underline{r}}{dt} + \underline{r} \cdot \frac{d\hat{\omega}}{dt} = 0 \qquad \because \frac{d\hat{\omega}}{dt} = 0$$

$$\Rightarrow \frac{d}{dt}(\hat{\omega} \cdot \underline{r}) = 0$$

Integrating: $\quad \hat{\omega} \cdot \underline{r} = \text{Constant} = h \text{ (say)} \longrightarrow ②$

This is an equation of plane perpendicular to $\hat{\omega}$ and at distance h from O. Hence the locus of p lies on this plane.

Thus the locus of p must be intersection of this plane with sphere in ①

Now the intersection of plane and a sphere is always a circle.

Therefore locus of P is a circle.

If v is the velocity of P, then

$$v^2 = \underline{v} \cdot \underline{v}$$

$$= (\underline{\omega} \times \underline{r}) \cdot (\underline{\omega} \times \underline{r})$$

$$= \begin{vmatrix} \underline{\omega} \cdot \underline{\omega} & \underline{\omega} \cdot \underline{r} \\ \underline{r} \cdot \underline{\omega} & \underline{r} \cdot \underline{r} \end{vmatrix}$$

$$= \begin{vmatrix} \omega^2 & \underline{\omega} \cdot \underline{r} \\ \underline{\omega} \cdot \underline{r} & r^2 \end{vmatrix}$$

$$v^2 = \omega^2 r^2 - (\underline{\omega} \cdot \underline{r})^2$$

putting $r^2 = a^2 \quad \underline{\omega} \cdot \underline{r} = h$

$$v^2 = \omega^2 a^2 - \omega^2 h^2 = \omega^2 (a^2 - h^2) \rightarrow (3)$$

If H is the foot of $\perp$ar from O to the plane given by (2), then H is the centre of the circle which P describes. Also

By pythagorian rule

$$OP^2 = OH^2 + HP^2$$

$$HP^2 = OP^2 - OH^2$$

$$= a^2 - h^2 \qquad \because P \text{ lies on sphere (1)} \quad \therefore OP^2 = a^2$$

$$\because H \text{ lies on plane (2)} \quad \therefore OH = h$$

using this in (3)

$$v^2 = \omega^2 HP^2$$

This constant as HP is constant and equal to $\sqrt{a^2 - h^2}$

Thus P describes a circle with constant velocity

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Instaneous Axis of Rolation and Instantaneous Centre of Rolation

If a point O is fixed in a rigid body has velocity $\underline{v}_0$ and if $\underline{\omega}$ is the angular velocity of the body, then the equation of central axis is

$$\underline{p} = \frac{\underline{\omega} \times \underline{V}_0}{\omega^2} + \lambda \underline{\omega}$$

where λ is a variable scalar parameter and $\underline{p}$ is the p.v of any point P on the central axis w.r.t O. The velocity of P is given by

$$\underline{V}_P = \underline{V}_0 + \underline{\omega} \times \underline{p}$$

$$= \underline{V}_0 + \frac{[(\underline{\omega} \cdot \underline{V}_0)\underline{\omega} - \omega^2 \underline{V}_0]}{\omega^2}$$

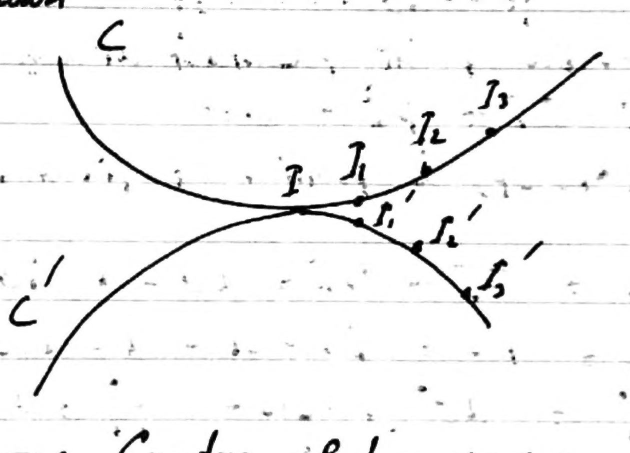
$$= \frac{(\underline{\omega} \cdot \underline{V}_0)\underline{\omega}}{\omega^2}$$

This shows that if $\underline{\omega}$ & $\underline{V}_0$ are non-zero, then $\underline{V}_P$ can only be zero if $\underline{\omega}$, $\underline{v}_0$ are perpendicular. In this case, every point on the central axis is instantaneously at rest and central axis in this case is called instantaneous axis of rotation; since the body as a whole is at that instant rotating about the central axis. If the two vectors are not perpendicular such a line can not be found.

Space Centrode and Body Centrode

The notion of instantaneous axis of rotation is especially important in two dimensional dynamics when a rigid Lamina is moving in its own plane. In this case velocity $\underline{v}_o$ of very point o of the lamina is at right angles to the angular velocity $\underline{\omega}$ since direction of $\underline{\omega}$ is normal to the plane of lamina. Thus for such a case the instantaneous axis of rotation will exist and its point of intersection with with the plane of the lamina is termed as the instantaneous centre of rotation of the Lamina

As the motion of the lamina proceeds, this centre, I, will trace out a locus fixed in space and this locus is called the space centrode and one locus is traced out by I in the body and is called the body centrode. The motion of lamina is equivalent to the rolling of the space centrode C on the body centrode C′ as shown



Instantaneous Centre of Lamina

If a lamina moves in its own plane so that its angular velocity is not zero, then there is a point fixed relative to lamina with zero velocity. This point which may change from instant to instant is

called the instantaneous centre and it need not lie on the Lamina.

Problem # A circular disc of radius a, and centre C rolls without slipping along OX and in the plane of YOX. If P is a point fixed on rim of the disc, then prove that when radius through P has rotated through angle θ, then

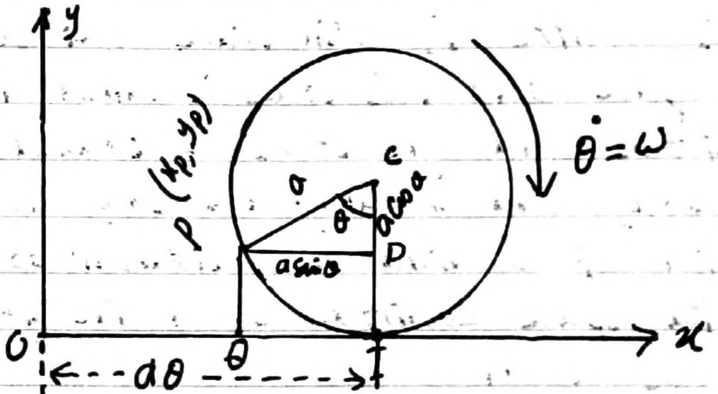
$$x_p = a(\theta - \sin\theta) \quad y_p = a(1-\cos\theta)$$

and

$$\dot{x}_p = a\dot{\theta}(1-\cos\theta) \quad \dot{y}_p = a\dot{\theta}\sin\theta$$

Also find instantaneous centre and space, body centrodes #

Sol #.



Let I be the point of contact at any time t

$\because \; OI = \widehat{PI} = a\theta$

If (x_p, y_p) is position of fixed point P at time t, then

$$x_p = OI - QI = a\theta - a\sin\theta = a(\theta - \sin\theta)$$

$$\left.y_p = PQ = IC - CD = a - a\cos\theta \right\} \rightarrow (1)$$

$$\Rightarrow \quad \dot{x}_p = a\dot{\theta}(1-\cos\theta \quad \dot{y}_p = a\dot{\theta}\sin\theta \rightarrow (2)$$

which shows that $\dot{x}_p = 0, \dot{y}_p = 0$ when $\theta = 0$ i.e. the point I is instantaneous at rest. Thus I is instantaneous centre. The body centrode is the rim of the disc, the space centrode is line OX. Since I is at rest, the velocity of any point P on the rim is $\dot{\theta}(PI)$ in the direction $\perp$ ar to PI in the sense of increasing θ

Problem # A rigid Lamina is moving in its own plane and one point A in it has velocity $\underline{u}$ relative to a fixed origin O. If $\underline{\omega}$ is the angular velocity of the Lamina, show that the point P in it has velocity $\underline{u} + \underline{\omega} \times \underline{r}$ relative to O. ~~Show that the point P in it~~, where $\underline{r} = \overrightarrow{AP}$. Hence or otherwise prove that the position vector $\underline{r}'$ of the instantaneous centre I of the lamina relative to A is given by $\underline{r}' = \dfrac{\underline{\omega} \times \underline{u}}{\omega^2}$

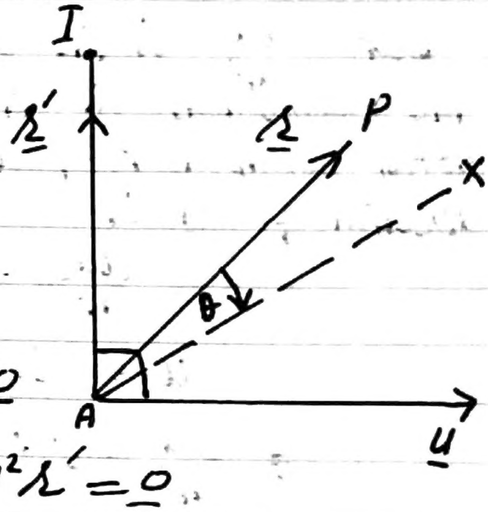
Show also that the acceleration of the particle in the lamina which instantaneously coincides with I has a component $\frac{du}{dt} - \left(\frac{u}{\omega}\right)\left(\frac{d\omega}{dt}\right)$ parallel to $\underline{u}$

Sol# The velocity of I

$= \underline{u} + \underline{\omega} \times \underline{r}'$

$\because$ I is instantaneously at rest

$\therefore$ $\underline{u} \times \underline{\omega} \times \underline{r}' = \underline{0}$



$\Rightarrow$ $\underline{\omega} \times \underline{u} + \underline{\omega} \times (\underline{\omega} \times \underline{r}') = \underline{0}$

$\underline{\omega} \times \underline{u} + (\underline{\omega} \cdot \underline{r}')\underline{\omega} - \omega^2 \underline{r}' = \underline{0}$

Since $\underline{\omega}$, $\underline{r}'$ are at right angles, $\underline{\omega} \cdot \underline{r}' = 0$ and so

$$\underline{r}' = \frac{\underline{\omega} \times \underline{u}}{\omega^2}$$

Relative to A, the acceleration of P has radial and transverse components $\ddot{r} - r\dot{\theta}^2$ and $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ where θ is the angle which AP makes with a direction AX chosen to be fixed in space. These components are $\ddot{r} - r\omega^2$ and $r\dot{\omega} + 2\dot{r}\omega$

when P coincides with I, then

$$\underline{r} = \underline{r}'$$

so that component of acc of I relative to A is transverse component $r'\dot{\omega} = r'\alpha$ in the direction of $-\underline{u}$, since $\dot{r}' = 0$, the velocity of I relative to A is perpendicular to AI

Now

$$u^2 = \underline{u} \cdot \underline{u}$$

$$\Rightarrow 2u \frac{du}{dt} = 2\underline{u} \cdot \frac{d\underline{u}}{dt}$$

$$\frac{du}{dt} = \frac{\underline{u}}{u} \cdot \frac{d\underline{u}}{dt}$$

$$= \hat{u} \cdot \frac{d\underline{u}}{dt}$$

But $\frac{d\underline{u}}{dt}$ is the acc of A in space and $\hat{u} \cdot \frac{d\underline{u}}{dt}$ the component in the direction of $\underline{u}$. Hence total component of acceleration of I in the direction of $\underline{u}$ is

$$\frac{du}{dt} - r'\dot{\omega} = \frac{du}{dt} - r'\frac{d\omega}{dt}$$

$$u = r'\omega \Rightarrow r' = \frac{u}{\omega}$$

$$\Rightarrow \frac{du}{dt} - r'\dot{\omega} = \frac{du}{dt} - \left(\frac{u}{\omega}\right)\frac{d\omega}{dt}$$